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A

TREATISE

ON

Descriptive Geometry,

FOR THE USE OF THE

CADETS OF THE UNITED STATES MILITARY ACADEMY.

BY C. CROZET,

PROFESSOR OF ENGINEERING IN THE ACADEMY.

PART I.

CONTAINING THE ELEMENTARY PRINCIPLES OF DESCRIPTIVE GEOMETRY, AND ITS APPLICATION TO SPHERICS AND CONIC SECTIONS.

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NOTICE.

The second part of this work, which contains the conclusion of Descriptive Geometry, and its applications to perspective, shades and shadows, and dialling, will be published as soon as sufficient encouragement has been obtained. Subscriptions will be received by the following booksellers:

Wells & Lilly, Boston.
Goodrich, Eastburn, Gilley, and Wiley & Halsted, New-York.
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The price of the second part will probably exceed that of the first, on account of its containing a greater number of plates; it is, however, expected, that it will not be above three dollars to subscribers.

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INTRODUCTION.

This Treatise on *Descriptive Geometry* was undertaken at the request of the Department of War, for the use of the Military Academy, where this new and useful branch of the Mathematics, having been taught for the last four years, with the greatest success, has become a fundamental study, necessary for the understanding of many important branches in the course of instruction.

Although the methods and principles of Descriptive Geometry cannot be said to be altogether new, this science ought nevertheless to be considered as a discovery of the celebrated *Gaspard Monge*, Count of Peluse. To him was reserved the honor of reducing, to a few elementary principles, the geometric constructions of which the arts were in possession. From these he formed a new science, which, by its simplicity, and general application, renders it easy for Engineers to comprehend a number of graphic methods, discovered separately by men of superior genius, and employed blindly by practitioners, who, being satisfied with copying their predecessors, confined their graphic knowledge to a very limited number of problems, and consequently were incapable of executing any thing that departed from their accustomed routine. Hence, it often happened, that instead of rendering graphic constructions subservient to the proper execution of a project, this last was so modified as to yield to the methods of practice.

Each branch had its own particular practical constructions; the most of them very limited, and all badly demonstrated; the same principles were seen to exist in all, but disguised and embarrassed by technical words, so that each

of the arts to which they could be applied, required a long and laborious study. It was, therefore, nearly impossible for an Engineer to understand perfectly all the secondary branches of his art; and thus he often found himself forced to depend on the workmen that he employed, to whose ideas and methods he was obliged to conform.

Monge, at the age of twenty, employed as a Draughtsman at the French School of Engineers, was not slow in perceiving that the graphic constructions there taught, in the different courses, were all reducible to the elementary principles of Geometry. He turned his attention, without delay, to furnishing the Engineers with general methods applicable to every case that could occur in practice, and freed from the shackles of empiricism. Descriptive Geometry was then the result of his earliest researches, and this discovery would alone have been sufficient to give him a distinguished place among the learned men of his time.

Notwithstanding the opposition of the old professors in each branch, the Students of the School of Engineers, disgusted by the obscure and incomplete manner in which they were taught Perspective, Architecture, Carpentry, Fortification, and the Constructions of Machines, applied themselves to learn the principles of Descriptive Geometry; and, proving by their success, of what advantage it was to them, finally compelled the chiefs of the institution to order its adoption into the course of studies.

Descriptive Geometry, strictly speaking, is only an art. It teaches the graphic construction of problems belonging to Geometry of three dimensions.

In plane Geometry the solution of a problem does not differ from its construction: let it be required, for instance, *to describe a circle through three given points*: we know that, by drawing perpendiculars through the middle of the lines that join these points, the point of mutual intersection is the centre of the circle required.

But questions in Higher Geometry have a different character. If, for example, it be proposed, as a problem, *to pass*

a sphere through four given points; the principles of Geometry teach us, that the centre is found by dividing equally each of the lines, that join respectively each two of the points, by a perpendicular plane; and that the common point of intersection of these planes will show the centre, and, of course, the radius of the sphere.. But this solution, although conformable in principle, to that of the preceding problem, cannot, like that, be constructed at once, because the figures employed in obtaining it are not situated in the same plane: it becomes, therefore, necessary to have recourse to operations susceptible of being performed on paper; this requires the employment of conventional methods:

1st. To fix the respective situations of the given points and lines.

2d. To furnish the means of passing regularly from one part of the operation to another, so as to determine successively, those results which cannot be obtained all at once as in Plane Geometry.

Thus, then, common Geometry shows the properties of lines and surfaces; and Descriptive Geometry, availing itself afterwards of these elementary principles, describes the figures of bodies, measures their dimensions, and determines by regular drawings, the results of these combinations, the bases of which are furnished by pure Geometry: the one is an operation of the mind, the other its graphic translation.

These two species of Geometry cannot, therefore, subsist without a mutual dependence upon each other: in fact, of what use would the principles of common Geometry be, if we had not graphic methods of constructing the results to which they lead us? As for Descriptive Geometry, the progress made in it will evidently be proportional to the degree of knowledge before acquired in Theoretical Geometry. It is however worth remarking, that in the more common cases of practice, there is only need of the very elements of this last science; for, the Geometrical figures employed in the arts are generally very simple.

The doctrine of perspective and of shadows, for instance,

requires nothing but some of the most simple problems of the cone and cylinder, which are laid down in the first pages of this work. This very agreeable and useful subject is therefore only a simple corollary of some of the elementary propositions of the new Geometry, and may henceforth be comprehended with the greatest ease by Artists. Every one must have remarked how imperfect the most voluminous writings on this subject are: the author flatters himself, that his Treatise will at least have the advantage over them by its brevity.

In Carpentry, and the construction of Arches, it is not often necessary to make use of any other figures than planes, cones, cylinders, and spheres: they consequently require nothing but a knowledge of the same principles as perspective; and this observation holds good with regard to the other graphic arts.

It may however be asked, if this simplicity of the primitive forms at present used, is not owing in a great degree, to the imperfection of our Geometric knowledge. Without attempting to decide this question, we will content ourselves with remarking, that instead of the obscure and limited problems formerly made use of, we are now possessed of new and fruitful methods, which are capable of being applied to all kinds of combinations that circumstances can require, or the mind conceive. We ought, therefore, from their application, to expect the greatest perfection in modern works: and this, not only from the facility we acquire in translating our projects into *graphic language*, but still more from the habit which the simplicity of the processes in Descriptive Geometry gives us of conceiving the forms of bodies, and deducing from their ideal representation the most abstract truths of Geometry. In fact, by studying this new science, the mind becomes accustomed to figure to itself lines and surfaces, arranged and combined in every possible manner. This action of the imagination increases its strength and its accuracy, so that it has been said, with truth, that Descriptive Geometry was the Science of the Engineer. The remarkable clear-

ness of its operations, its peculiar character of evidence, the faculty which it gives of beholding mentally the moving representation of Geometrical figures, cannot fail of being extremely useful in practical mechanics; as the perfection of machines depends principally upon the ease with which we conceive and combine their elements. The applications which have been made of these new methods, to some of the most important theories of machines, are sufficient to show what assistance it will afford to this branch of science: it is, however, remarkable, that as yet, much less has been done in this application of it, than in any other.

As for the construction of arches, of bridges, as well of stone as of wood; the formation of roads and canals; the building of ships, the perfection of which depends so much upon that of graphic processes, great progress has been made in them by the help of Descriptive Geometry, which sheds a light upon all the conceptions of an Engineer, that can never give place to obscurity and uncertainty.

The impossibility of embracing at one view the whole of a plan, has frequently constrained an Engineer to begin the construction of his work before the project of it was completely organized. Hence, have often resulted errors in the details, unthought of difficulties in the execution, and useless expences and faults in the assemblage of the parts: but since the new graphic methods have been brought into use, there is no longer reason to fear those mistakes so common in ancient constructions. The applications of this science, that have recently been made to fortification, and particularly to *defilement*, have given to this branch of the Military Art, a perfection and an extent of means, that have very much contributed to the superiority of modern fortification.

The same may be said of the construction of ships: the scientific Carpentry applied to this purpose, demands the most delicate graphic operations, and has received within the last few years the greatest perfection, owing almost entirely to the researches of the celebrated Engineer, Charles Dupin, who at present holds so high a rank in the Academy of Sciences of Paris.

The advantages of Descriptive Geometry are also extended to what may be more strictly called, the Mathematical Sciences; some of the most beautiful theories of which, owe their developement to it. The application of Algebra to Geometry, that elegant analysis, which has thrown such light upon the important subject of Partial Differentials, and which has immortalized the name of Monge, requires the use of Geometric conclusions, at which it would have been difficult to arrive without the aid of our science. There are even but few problems of Plane Geometry itself, to which it cannot be made applicable. This will be evident from the Treatise on Conic Sections, which follows the first part of our work. In a word, with the aid of a small number of the elementary propositions of Geometry, this science possesses an almost infinite variety of means, by which we may arrive at the solution of the most difficult problems. It requires but a few weeks study to be sufficiently understood: it advantageously replaces the common modes of practice, the long and laborious study of which it renders unnecessary; at the same time that it gives us the immense advantage of treating with equal ease new combinations or unforeseen cases.

DIVISION OF THE WORK.

The present work is divided into two parts: the first contains the Elementary Principles of Descriptive Geometry; that is to say, those which are most commonly applicable to the Arts, and which require only a knowledge of the most simple parts of Geometry. At the end of this are its applications to Spherical Trigonometry, and a Treatise on Conic Sections.

The second part, the publication of which will shortly follow that of the first, will contain those propositions of a higher order, which form the complement of the Science. These will be found necessary for the understanding of many

practical operations of a more scientific nature; and will particularly be indispensable to those who have to organise important works; as, by means of it, they will be enabled to surmount unexpected difficulties, which would otherwise often stop their progress, should their knowledge not be proportionate to their conceptions. At the end of this second part, there will be a complete Treatise on Shadows and Perspective, and some applications to Dialling.

The Treatise on pure Descriptive Geometry, contains only general and fundamental propositions, by means of which, strictly speaking, we can resolve all questions in high Geometry: it is, notwithstanding, evident that the data of these problems can be varied in such a manner as to present particular cases, some more difficult, while others are more susceptible of simplification. A complete treatise should contain a great number of examples, to serve as exercises for beginners, and to give them an aptitude in graphic operations; but the author has thought that it would be more advantageous to unite the applications of Descriptive Geometry, with the Elementary Problems, in this Treatise; because, by this means, particular cases would present themselves in their proper place, in a manner more evident and more useful to the reader; who, instead of studying a series of dry and detached propositions, which he might afterwards have to apply to some particular graphic art, will thus be able, at once, to comprehend both the one and the other. On this account the doctrine of Shades and Perspective is added to the second part.

As for Conic Sections, demonstrated by the methods of Descriptive Geometry, they may be considered as a proper study for exercise in its principles. The great use which is made of these curves in graphic operations, unites them closely to the science which treats of these operations, and furnishes means of discovering and demonstrating their properties, equally simple, as certain and methodic. And we must here remark, that the most of the Treatises on Conic Sections demonstrate them by a chain of propositions, in ge-

neral very complicated, and which, depending upon each other, require so strong an effort of memory to retain them, that they are really of very little use in practice. It rarely happens, in fact, that these propositions can be made applicable to particular cases; it often becomes necessary to modify them; and, oftener still, is there need of referring to some new properties, which it is very difficult to discover, by means of theorems, whose demonstrations are so intricate.

The very name of these curves shows that they are obtained by the intersection of surfaces: why, then, may we not there search for their properties? why, may we not, by a direct method, deduce from the operation itself, by which they are obtained, the means of arriving at a knowledge of these same properties? To attain this end, the Treatise on Conic Sections has been written: almost all the demonstrations are new; they might have been all, perhaps, deduced immediately from the cone, but some would then have been too complicated; besides, if it be advantageous to obtain them thus from their common source, it is sometimes useful to infer one from another. The number of propositions that belong to this family of curves is almost infinite; it has not, therefore, been thought worth while to offer any but such as are commonly useful; but, at the same time, care has been taken to render the method of obtaining them perfectly evident. We do not convey to another an actual knowledge of a science, by merely teaching him a certain number of its principles, but by shewing in what manner those principles are primarily obtained, and thus giving him a key to all his future researches. Conic Sections, considered in this light, have, moreover, the advantage of serving as an introduction to the investigation of the numerous curves, which are produced by the intersections of surfaces in general.

It may not, perhaps, be improper to recommend, in addition to the study of Descriptive Geometry, the graphic construction of its different problems; in which case, it will be well to vary the data. There is in graphical operations a

certain skill in the execution and a judicious arrangement of parts, which cannot be acquired but by long practice. It is, above all things, important that the Student should early accustom himself to draw with neatness and exactness, the curves necessary to any operation; since curves are very frequently employed, and ought even, at times, to be preferred to straight lines and circles. For, although it may not be possible to draw a curve with the same exactness as a straight line or a circle, it is nevertheless evident, that it is better to determine a point by the intersection of two curves, than by a great number of straight lines and circles. When Geometers propose to solve a problem, they generally exclude the use of curves; but, in graphic constructions, we should be wrong not to avail ourselves of them, in preference to a great complication of more simple lines, especially when these cut each other at too acute an angle: again, we must employ all the resources which the fruitfulness of Descriptive Geometry furnishes, to avoid by some modification of the general solution, this last disadvantage, which is one of the principle sources of inaccuracy in graphic operations. In fact, of two lines having a certain breadth, it is very difficult to find exactly the place of intersection, which will always appear rather longer than a point ought to be; therefore, we ought to use, as much as possible, those lines that form nearly a right angle. As for the drawing of curves, it is not so difficult a matter as first attempts would make it appear: a mathematical curve has always a kind of regularity which the practised eye easily seizes upon; and, besides, the surfaces, by the intersection of which it is formed, will, almost in every case, help to investigate its direction, and to determine its most characteristic points, such as its cusps, inflexions, maxima and minima, as well as its asymptotes and tangents; a knowledge of which is of singular assistance in the description of a curve.

The illustrious author of Descriptive Geometry is no more; he died in 1818, at the age of 72 years. Besides being

conspicuous from his scientific works, he is well known as the founder of the Polytechnic School, the plan of which has received the sanction of time and of nations. The students of this school will always cherish his memory. They will never forget the kind and winning manner with which he gave them his lessons; appearing unconscious of his immense superiority, he never perceived the admiration with which he inspired his hearers. He taught with the simplicity that always characterizes true genius. Entirely occupied with the interest of his pupils, he answered with unalterable good nature to every question which they proposed, varying and repeating his explanations, until he perceived that they were understood. By daily proofs of his attachment, he knew how, to insure their love and their respect. They will always recollect with pleasure, that neither his duties as president of the French senate, nor his occupations as a member of the institute, could prevent him from continuing to give them, with the most scrupulous punctuality, his admirable lectures. His amiable affability was never diminished by a thought of the distinguished rank which he held. Affectionate and disinterested, he constantly gave up his salary to the treasurer, for the benefit of those of his pupils who were without fortune. It is to be regretted that this great man, with his almost universal knowledge, has written so little; but how could we blame him for this, when we call to mind the answer he gave to some friends, who were persuading him to publish a course of mathematics: *Bezout, said he, has left a widow who has no other fortune but her husband's works; and I do not wish to deprive of bread the relict of a man who has rendered such services to science and his country.*

At the beginning of the French revolution, Monge showed himself in favour of such a change in the government, as would allow merit to obtain the distinction heretofore bestowed on rank. At this epoch, he was entirely occupied in contributing, as much as lay in his power, to the glory of his country; and showed himself a republican only by disinterested and praise-worthy actions. Animated by the

purest patriotism, he devoted a considerable portion of his fortune to the relief of the soldiers of General Macdonald's division, who, at their return from Italy, were in want of every thing.

It was with the same desire of being useful to his country, and to the sciences, that he placed himself at the head of that society of learned men who accompanied Napoleon to Italy and Egypt; and the arts are indebted to him for the preservation of some of their most beautiful chef d'œuvres.

The republican principles of this great man brought upon him the persecutions of a government hostile to liberal ideas. The first act of injustice which was shown to him was the erasure of his name from the rolls of the institute: this and other injuries he received, so wounded his high and generous spirit, as to cause him to fall into a gloomy melancholy that failed not soon to conduct him to the tomb. His obsequies were not attended with any parade; but, if the fear of offending the government, for one moment stopped the expressions of gratitude and admiration, it could not long repress them. His pupils and his venerable colleagues, the Berthollets, the Chaptals, the Vauquelins, the Laplaces, &c. spontaneously repaired the next day to his grave, and there, calling to each others recollection all the fine traits of his character, and the virtues of his life, mixed their tears in sincerely lamenting the loss of a man, at once so good and so illustrious, whom they all had loved. On separating, they placed a crown of laurel on his tomb, and resolved to erect a monument, worthy of him, worthy of the nation that was so much indebted to him; and, thereby, to avenge the injury he had received from the government.

Descriptive Geometry.

CHAPTER I.

PRELIMINARY DEFINITIONS AND PROPOSITIONS.

1. In the numerous applications of Geometry to the arts, it is often necessary, not only to represent objects in their various shapes and respective situations, but also to measure and determine their parts with accuracy, and to deduce from their form and position results required by the nature of the operation to be performed.

But it is evidently impossible to represent on the plane of a drawing, with their two proportions, bodies or figures, whose parts are neither in, nor parallel to, the same plane. And consequently some auxiliary mode of representation and solution must be used, to perform with accuracy, the graphical operations required.

The science which teaches the general principles of these graphical solutions called is Descriptive Geometry.

Its objects are, therefore,

1st. The representation of bodies. 2d. The consequences to be deduced from their properties.

2. It is evident that the methods used to represent regular ob-

Descriptive Geometry.

CHAPTER I.

PRELIMINARY DEFINITIONS AND PROPOSITIONS.

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jects must be both correct and simple; in order that the graphical results may be easily obtained: these advantages are found in the use of *projections*.

3. A *projection* is, in general, the figure formed on a given surface, by the intersection with a system of lines, drawn to the different points of an object.

4. *Descriptive Geometry teaches the use of projections.*

5. The principal kinds of projections are the *Orthogonal projection* and *perspective*.

The *Orthogonal projection* is determined by a system of lines perpendicular to the same plane. It is generally employed when exact dimensions are required.

A *Perspective* results, in general, from the intersection of a surface, commonly a plane, by a system of lines drawn from the same point to the different parts of the objects to be represented.

This kind of projection is generally known; its principles and operations are very easy consequences of the methods of Descriptive Geometry, to which it belongs, more as an application than as a mode of solution: for, the object of perspective is to represent things as they are seen from a certain point; and, consequently, it can no more give an exact idea of their true dimensions than the actual view of them. It is rather a pleasing and lively delineation, than a correct Geometrical projection; and is, in general, used as an agreable auxiliary of taste and invention; but very seldom when accuracy is required.

The Orthogonal projection is, therefore, almost exclusively employed in the operations of Descriptive Geometry; as, not only the most accurate, but the most simple method of determining the position of points, and consequently of lines and surfaces.

This will appear evident by considering, that the position of a point cannot be fixed otherwise than by its relation to some known objects; and that these objects must be simple, in order to be of an easy practical use. Hence the determination of a point by means of its distances from three known planes, seems to be the most convenient of the various methods which can be employed.

For, let us suppose that the point is known to be at a distance of two feet from a plane A; it will evidently be contained in a plane parallel to it, at this given distance.

If, besides, it must be at a distance of three feet from a second plane B, a plane parallel to B will also contain it: and, therefore,

it is one of the points of the right line in which those two first planes intersect.

Lastly; another plane parallel to the third given plane C, will, by its intersection with this line, determine completely the position of the point.

This method of finding the position of a point, by means of its distances to three given planes, is employed in the application of Algebra to Geometry.

But, in the operations of Descriptive Geometry, where, not only the distances, but the projections of the point are given, two planes are sufficient to determine it; as will soon appear obvious. (8.)

PROJECTION OF A POINT.

6. The *foot* of the perpendicular let fall from a point upon a plane is the *Orthogonal projection* of the point on that plane: it will hereafter be called merely *projection*. a, a' (fig. 1) are the two projections of the point A on the planes LO and PL.

7. The planes upon which projections are made, are named *Planes of Projections*.

8. Let us now conceive two planes any ways inclined to one another: if the projection of a point in each one of them is known, this point is perfectly determined.

Fig. 1. for, if through the projection a, in the first plane LO, a perpendicular Aa be drawn to this plane, it will evidently pass through the point A: in like manner another perpendicular $a'A$ erected through the other projection a', to the second plane PL, will also contain this same point; which will, therefore, be perfectly determined by the intersection A of the two perpendiculars Aa, Aa'.

This is independent of the angle formed by the two planes: but if that angle was obtuse or acute the perpendiculars drawn from a point to each of them would be oblique to each other; which might sometimes be the cause of great errors. For this reason, and because the intersection of perpendicular lines is the most favourable to the correct determination of points, rectangular planes of projections are generally employed.

Besides, in consequence of the natural idea of perpendicularity

which results from it, it is customary to suppose one of the planes of projections *horizontal*, and the other *vertical*.

Moreover, because of the necessity of including both projections in the same drawing; (the horizontal plane being fixed) the vertical plane is always supposed, after having received all the projections, to have been brought down upon the horizontal plane by a revolution round its intersection with it. So that, in fact, the horizontal and the vertical projections are traced in the same plane, though the latter must always be conceived as standing at right angles with the former.

It is somewhat difficult, at first, to consider as vertical a plane which does not in fact differ from the plane to which it is perpendicular: a very simple way to accustom the mind to it, is, after having folded a sheet of paper, to raise one half of it perpendicular to the other, and then to open again the sheet entirely. Whatever might be drawn on the perpendicular part of it will remain the same when it is open; and consequently may as easily be conceived then, as when it was standing at right angles with its new position. After having studied the two or three first problems in this way, this difficulty will be so completely done away with, that it will be almost impossible to conceive the two planes of projections otherwise than at right angles with each other.

9. The line of intersection of the two planes of projections is called the *common intersection*, *the ground* or *base line*, and sometimes *the fundamental line*. (See *LM*. fig. 2. and 2 bis.)

As it divides the two planes, when they are made to coincide, it must always be drawn very distinct.

10. The present method of projection has also another advantage which is very convenient in practice, viz. that, *the two projections of the same point are situated, when the two planes of projections coincide, in a line perpendicular to their common intersection.*

Fig. 1. for, let a, a' be the horizontal and vertical projections of the point A: it is well known that a plane drawn through the perpendicular lines Aa, Aa' will be perpendicular to each of the planes of projections LO, LP; and consequently to their common intersection LM; which is reciprocally, therefore, perpendicular to the lines $a'C$, aC, in which the perpendicular plane $Aa\,Ca'$ cuts the two planes of projections.

But when the plane PL revolves round its intersection LM

with the horizontal plane, the line $a'C$, carried round with it in this motion, does not cease to be perpendicular to LM, and will still be so when the vertical plane PL coincides with the horizon tal one: the line $a'C$ takes then the position $a''C$. Therefore the two lines Ca, Ca'', passing through the same point C, and both perpendicular to LM, are in the prolongation of each other.

This is one of the fundamental propositions of Descriptive Geometry.

11. It would be equally true if the planes were oblique to each other. But rectangular planes of projections are used in preference; not only on account of the advantage already mentioned, but also because the distance of a point to one of the planes of projections is measured, at once, by the length of the perpendicular drawn from its projection in the other plane, to the common intersection: that is $Aa=a'C=a''C$ and $Aa'=aC$, a property oblique planes do not possess, and which will be found very convenient in practice.

12. These are all the preliminary remarks which it is necessary to understand, in order to obtain a correct idea of the projections of a point. We will conclude them by this last observation;

That: *when a point is situated in one of the planes of projections, it is its own projection in that plane. Its other projection, (when the planes are perpendicular,) is evidently in the ground line.*

PROJECTIONS OF A RIGHT LINE.

13. The projection of a right line is another right line. For, the perpendiculars let fall from every point of the line upon the plane of projections are all parallel, and therefore in the same plane.

The intersection of this plane with the plane of projections is the *projection of the line*: the plane which projects a line is the *projecting plane* of this line,

14. Fig. 2. A right line, like a point, is perfectly determined by its two projections: for, if through these projections, two planes are drawn perpendicular to the respective planes of projections, they will each contain the line, and therefore determine it by their intersection.*

* However, the two projections of a line would be insufficient, if the line was in a plane perpendicular to both planes of projections; because in this case

15. In many graphical arts, and especially in architecture, the horizontal projection is called *geometral plan* or simply *plan;* and the vertical projection, *Elevation*, when it represents exterior parts; but, when it intersects the building or other object to be drawn, it is then called *section* or *profile*.

Before proceeding any farther in this elementary introduction, it will not be amiss to illustrate the preceding principles by the solution of a few problems; in order that they may become quite familiar, before a greater complication of new definitions and considerations is introduced.

SOLUTIONS

Of some Elementary Problems on the Right Line.

PROBLEM I.

The two projections of a line being given, to find the points where this line (supposed indefinite) passes through the planes of projections.

16. Let us, in the first place, find the point where the line pierces the horizontal plane.

Fig. (2 & 2 bis.) It is evident that this point is somewhere in the horizontal projection of the line: for, as it is a point of the horizontal plane, it must coincide with its own projection.(12)

But the vertical projection of this same point is somewhere in the common intersection;(12) and, at the same time, in the vertical projection of the line which contains it; it will consequently

there would be but one projecting plane. It would then be necessary to fix the position of the line by some additional condition; such, for instance, as its projection on a third plane.

When a line is parallel to one of the planes of projections, its projection in the other is parallel to the common intersection of the two planes.

If it is perpendicular to one of them, its projection in it is a point.

be determined by their intersection. This vertical projection d of the point being known, the horizontal projection of the same is very easily deduced from it: for, the projecting line Dd must contain (11) the point sought, since it is the line by which it would be projected into the vertical plan. This point, being also in the horizontal projection of the line, will be determined by its intersection D with the line of projection Dd.

17. Though we would not recommend the use of figures in perspective, we have represented the whole of this first operation in perspective in the 2d. fig. To facilitate the study of the graphical ~~the~~ solution of it.

Fig. 2. (bis.) represents the construction with the same letters used in fig. 2, when the vertical plane $NPLM$ has been made to coincide with the horizontal plane $LMOQ$, by a revolution round the common intersection LM.

The first case is when the whole of the line is seen.

The second, when the line pierces the horizontal plane behind the vertical plane of projections.

The next supposition is that of the line's passing through the horizontal plane to meet the vertical plane below.

The fourth and last position is when the line passes both below the horizontal plane and behind the vertical one.

It is necessary to understand well these different situations of the line, as they will frequently present themselves.

Dotted lines are here, and will always be, used to designate those parts that are situated below the horizontal plane and behind the vertical one.

In general, those lines only that are the object of the operation are drawn full, when seen. The different auxiliary lines used in the construction of the solution are made up of short lines and dots in various ways, taking care to draw, of the same kind, all those that are used for the same object. This will considerably lessen the confusion, which would otherwise result from a great number of lines belonging to the different parts of an operation, and crossing each other in every direction, without any thing to distinguish them from each other, and to know at once what part of the solution they were intended for.

PROBLEM II.

The two projections of a line being given to find the angle it makes with either of the planes of projections.

18. It has been taught in the elements of geometry, that the angle formed by a line and a plane is measured in the perpendicular plane passing through the line; or, in other words, is the same as the angle contained between the line and its projection on the given plane.

Fig. 2 & 2 bis. Let us therefore suppose that the angle required is that which the line makes with the horizontal plane: as this angle is included between the projection of the line and the line itself, if the projecting plane is conceived to revolve round its base Db, until it comes in contact with the horizontal plane, the given line will by this operation become a line in the horizontal plane; and then its angle can be measured.

In order to understand this operation perfectly, the line Db must be considered as a fixed axis, or *hinge*, round which the projecting plane BbD revolves. It is evident, that, in this rotation of the plane, every point of the axis will remain fixed; and that every other point, like B, will fall at a distance $B'b$ from this axis, equal to its distance from the same line when the plane was standing: but this distance is evidently the vertical height Bb of the point above the horizontal plane; consequently, if the perpendicular bB' to the horizontal projection bD be made equal to the altitude bB, then will B' be a point of the given line, after it has fallen upon the horizontal plane.

But the point D where this line meets the same plane, being in the axis of rotation, must remain fixed; and, therefore, it will, together with the point B', determine the new position $B'D$ of the line. $B'Db$ is then the angle required.

PROBLEM III.

The projections of two points being given to find the length of the line which joins them.

19. If the points were at an equal distance from one of the planes

of projections, the line would be parallel to it; and consequently its real length would be measured by the length of its projection in that plane.

When the line is oblique to both planes of projections, its length is greater than that of either of its projections; but it may be deduced very easily from them.

Fig. 2 & 2 bis. For, it is then the hypothenuse *AB* of a triangle, the base of which *AC* is equal and parallel to its horizontal projection *ab*, and its altitude the difference *BC* between the projecting perpendiculars *Bb*, *Aa* of the extremities of the line.

This triangle may be constructed by drawing through the lower point *A* (fig. 2. bis.) the horizontal line *AC*; then will *BC* be the difference of the altitudes of the points *B* and *A*; because *Aa'* and *Bb'* (fig. 2.) are respectively equal to *Aa* and *Bb* (11). If now *A'C* be made equal to the horizontal projection *ab*, the hypothenuse *A'B* will be the length required.

The following solution may sometimes be easier in practice:

Knowing the point *D* where the line pierces the horizontal plane, draw to the horizontal projection *ab* the perpendiculars *aA'*, *Bb'*; make one of them *B'b* equal to *Bb'*, draw *DB'*; the intercepted part *A'B* is the true length of the line.

This operation is similar to the construction of the preceding problem. While the projecting plane *EDe* is revolving round the horizontal projection *De*, the vertical lines *Aa*, *Db* remain, in every position of the plane, perpendicular to the hinge *De*, and will yet be so when the plane coincides with the horizontal plane.

20. The same operation might be performed in the vertical plane, and must give the same result.

The solution of this problem is sufficient to determine the sides of any solid whatever bounded by planes, when its projections are given.

These first problems must be accurately drawn in the four cases above mentioned, in order that they may be thoroughly understood. The subsequent operations will appear much easier after having once obtained a clear idea of the fundamental propositions.

PROBLEM IV.

Through a given point to draw a line* parallel to another line given by its two projections.

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21. The projections of parallel lines are parallel: for they result from the intersections of parallel planes with the plane of projections.

Therefore, all we have to do is to draw through the projections of the point lines parallel with the respective projections of the given line. This requires no figure.

CHAPTER II.

OF THE TRACES OF A PLANE.

22. The surface of a plane being indefinite,† it would be impossible to represent it by projections: to a direct representation we must therefore substitute some method to determine the plane. The most simple way of doing it is, evidently, to fix by their projections the position of two lines of this plane. As one plane only can be made to pass through two lines, these two lines will completely determine the plane.

Among the lines which might be used for this purpose, the most convenient are the two in which the plane intersects the two planes of projections; they are therefore always employed to represent it.

We will call them the *traces* of the plane: each trace taking the name of *horizontal* or *vertical* according to the plane of pro-

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* By the word *line* we will hereafter designate the two projections by which it is determined, when no confusion can result from it.

† In the Elements of Geometry some limits must be assigned to planes, in order to render the figures used intelligible. But a plane may evidently be conceived as extended in every direction, to an indefinite distance: it is the idea which must be formed of a plane in all the operations of Descriptive Geometry.

jections which contains it. *VC* is the vertical, and *CH* the horizontal trace of the plane *VCH*. (fig. 3.) It is hardly necessary to mention that they always intersect each other in the ground line; since their point of intersection must be in each plane of projections.

23. If one of the traces is perpendicular to the common intersection, the plane itself is perpendicular to that plane of projections which contains the other trace: for, it passes through a line perpendicular to it. This latter trace is evidently the projection of the plane itself, and consequently of any figure described in it.

24. If, therefore, both traces are perpendicular to the common intersection, the plane is perpendicular to both planes of projections.

25. The two traces being parallel to the ground line, it follows that the plane is parallel to this line.

26. Lastly, if a plane has but one trace, (and in that case it must be parallel to the common intersection,) then the plane is parallel to the plane of projections which contains no trace.

These propositions are almost self-evident.

The elementary principles, which have been successively presented to the reader, are sufficient to solve all the questions relative to the right line and plane. We will, therefore, proceed to the examination of the most important of them.

Solutions of the principal Problems on the right line and plane.

PROBLEM V.

A plane being given by means of its two traces, to find the vertical projection of a point of this plane, when its horizontal projection is given.

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27. Fig. 3. Let *VA* and *HA* be the traces of the plane; and *P* the horizontal projection of the point.

Conceive through the point itself a line parallel to the horizontal trace *HA*; its horizontal projection must be a parallel *Pa* to

Pa

this trace ; and its vertical projection a line parallel to the common intersection (note No. 14), the position of which remains to be determined.

28. But the parallel, conceived to pass through the point, must be in the given plane, since it is parallel to its trace ; and, therefore, the point in which this line meets the vertical plane, being also one of the points of the oblique plane, will be found some where in the vertical trace *VA* of this plane : but it has been demonstrated (16) that this point is, at the same time, in the line *aB* perpendicular to the common intersection at the point *a*, where this latter is cut by the horizontal projection *pa* of the parallel ; consequently the point of intersection *B* of the vertical line *aB*, and of the trace *VA* is the point where the parallel to the horizontal trace pierces the vertical plane.

Draw then through this point *B* a horizontal line *BP*, which will be the vertical projection wanted.

But the vertical projection, corresponding to the horizontal projection given, is both in this line *BP*, and in the perpendicular line of projection *Pp* (10 :) it will, consequently, result from their intersection at *P*.

The same solution would answer if the vertical projection of the point were given instead of the horizontal ; since the names given separately to the planes of projections are arbitrary, and consequently cannot affect in the least the data of a problem in either of them. The construction in this supposition is represented in the figure, and may be used to verify the operation. What is said of one of them must in every case be understood as equally applicable to the other.

PROBLEM VI.

Knowing the two projections of a point situated in a given plane ; if the plane be made to revolve round its horizontal trace, it is required to find what shall be the position of the point in the horizontal plane, when the given plane comes in contact with it.

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29. Fig. 3. Every point of the given plane *VAH*, in its rotation round the horizontal trace *HA*, is always at the same distance from the axis of rotation, and consequently describes round it a circle whose plane is perpendicular to it.

But a plane perpendicular to the axis of rotation is, in the present case, a vertical plane : therefore, when the point P, p (that is; the projections of which are P and p) moves with the plane VAH round HA, it describes, in the vertical plane $p\ E\ P''$, a circle of which the centre is E.

The radius of this circle is the hypothenuse of a triangle, the base and altitude of which are $p\ E$ and PP'.

Lay off from P' to E' a line equal to ~~pE~~; and PE' shall be the hypothenuse required, which being carried from E to P'', gives the new position of the point. pE

30. If the parallel line pd, PD, had been previously drawn, an arc might be described from the point d with a radius equal to Pd; the intersection P'' of this arc with the perpendicular pEP'' would also determine the point.

For (19) PD, being the projection of a parallel to the vertical plane, is the true length of the line which joins, in the oblique plane, the given point with the point d of the horizontal trace; and this distance must be the same whatever may be the position of the plane; whence the foregoing construction.

This problem is of a very great practical use; it serves, as will readily be understood, to determine at once the real shape and dimensions of figures contained in a plane oblique to the planes of projections.

PROBLEM VII.

To draw, through a given point, a plane parallel to another plane given by its two traces.

Fig. 3. 31. VCH being the given plane, and P, p the point; any line passing through this point and parallel to the given plane is, of course, in the plane required: Consequently the line BP, pa, drawn parallel to the horizontal trace HC, aAC, (21, note No. 14.)* is in the new plane, the trace of which must pass through the point B, where that line pierces the vertical plane.

But the intersections of two parallel planes by a third plane are parallel lines; consequently the parallel VA, drawn through B, is the vertical trace of the plane required.

* Since the trace HC is in the horizontal plane, its vertical projection is the common intersection aAC (No. 14.) to which pa is drawn parallel.

For the same reason, and because the two traces must meet in the common intersection, the line HA parallel to HC is the horizontal trace of the same plane.

The construction of this problem might be made for the horizontal trace first, and would give the same result. In almost every case the correctness of the operations may be verified in this way.

PROBLEM VIII.

To find the intersection of two given planes.

Fig. 4. 32. Every point of the horizontal trace BD being in the first plane, and each point of the trace AD in the second, the point of intersection D of these two lines is at the same time in both.

In like manner, the intersection C of the two vertical traces is common to the two planes. Therefore the line of intersection of the two planes passes through the points D and C.

Project then the point D upon the vertical plane by the perpendicular Dd; and the point C into the horizontal plane by the perpendicular Cc; the lines Cd and cD will be the required projections.

The true length of the line would be found by describing the the arc DD' and joining CD' (19)

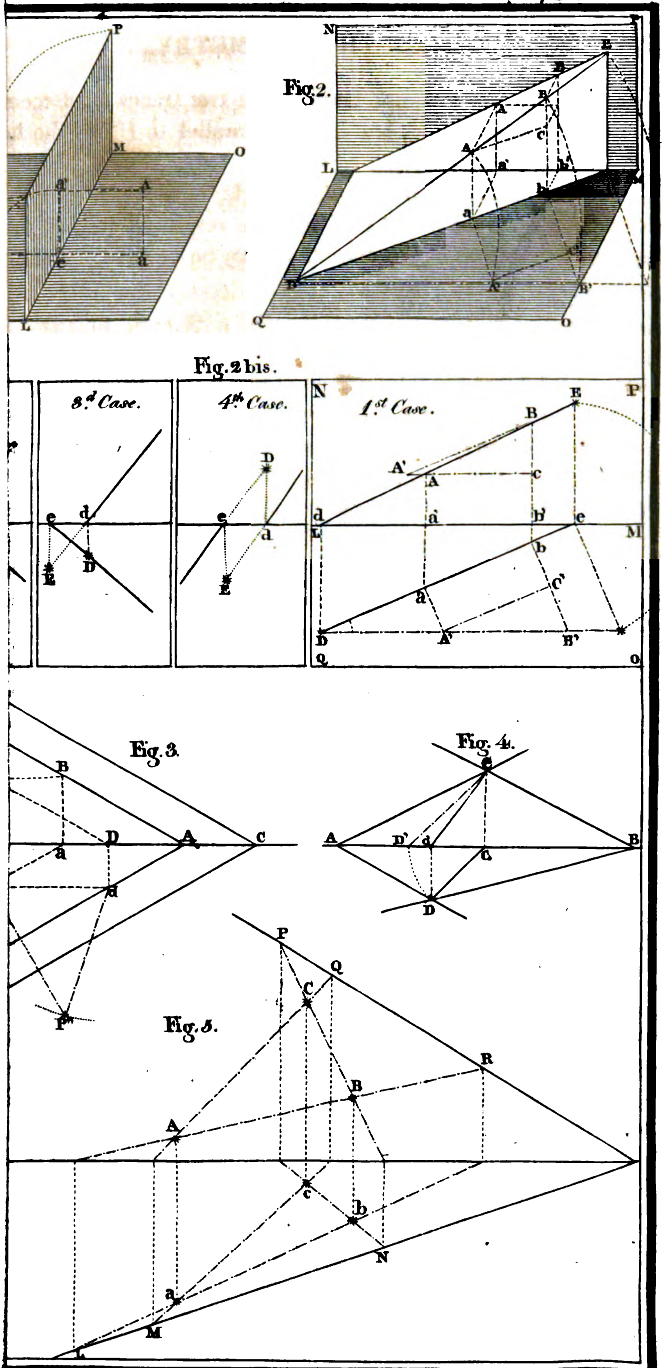
PROBLEM IX.

Having given three points, to construct the traces of the plane passing through them.

Fig. 5. 33. After having joined the three points by three right lines, find the points where these lines intersect the planes of projections (16); and then the lines which unite these last points must be traces required. (28)

In the figure; A, a; B, b; C, c are the points: the lines which connect them determine in the vertical plane the points P, Q, R; and in the horizontal plane the points L, M, N; therefore the lines PQR, LMN are the traces sought. The accuracy of the operation is proved when the different points found in the same

Plate 1.



plane of projections are in a right line, and the two traces meet in the common intersection.

34. If the lines did not intersect the planes of projections within the limits of the paper, other lines resting on them should be substituted. This is done by assuming* in each line a more advantageous point; the three points thus assumed are then used as the given points themselves, and must give the same result; since they are in the same plane with them.

PROBLEM X.

To determine the intersection of a line and a given plane.

35. Conceive any plane whatever passing through the line; this plane will intersect the given plane in a right line which, because it is in the auxiliary plane as well as the given line, shall cut it in a point: this point is the intersection of the line and the plane.

36. A plane always passes through a line when its traces contain the points where the line intersects the planes of projections; for, in that case, the line has two points in the plane. But, among the infinite number of planes that might be made to pass through the line, the most convenient, in the present case, are the projecting planes of the line.

Fig. 6. Let *BM*, *bm* be the projections of the given line; through the horizontal projection *bm* conceive the vertical projecting plane *C c m;* its vertical trace *Cc* will be perpendicular to the common intersection (23). then construct (by prob. viii.) the vertical projection *CF* of the intersection of the foregoing plane *Ccm*, and the given plane *CAD;* the point *P* where this line *CF* cuts the vertical projection *BM* of the given line is the point wanted.

The horizontal projection of the same is determined by the intersection of the projecting line *Pp*, and the horizontal projection *bm* of the given line.

A similar construction in the vertical plane would verify the result.

* In assuming a point in a line, it must be recollected that its two projections, which are situated in those of the line, must be contained in a perpendicular to the common intersection (10.)

THEOREM.

When a line is perpendicular to a plane, its projections are perpendicular to the corresponding traces of the plane.

37. For; Through the line conceive a vertical plane; it will be perpendicular not only to the horizontal plane, but also to the given plane; since it passes through a perpendicular to it: therefore it must be perpendicular to their intersection; that is to say, to the horizontal trace of the plane.

But the vertical plane drawn through the line contains the horizontal projection of it; to this projection, therefore, the trace must be perpendicular; since a line perpendicular to a plane is at right angles with every line in it.

38. The reciprocal proposition is equally true: that is, *when the projections of a line are perpendicular to the traces of a plane, the line and the plane are respectively perpendicular.*

For, if through each projection a plane be raised perpendicular to the plane of projections of the same name, the two planes thus determined will each be perpendicular to the corresponding trace, and consequently to the plane itself: therefore, the intersection of these two projecting planes, which is the line itself, must be perpendicular to that same plane.

PROBLEM XI.

Through a given point it is required to draw a line perpendicular to a given plane; and to find its intersection with the same.

Fig. 6. 39. Draw through the projections M, m of the point two perpendiculars to the respective traces of the plane; these perpendiculars will (by the preceding theorem) be the projections of the perpendicular to the plane.

The point where this line meets the plane is found by problem X.

PROBLEM XII.

To draw through a given point a plane perpendicular to a given line.

Fig. 7. 40. The directions of the traces of the plane are known to be perpendicular to the projections of the line: therefore it remains only to find their position.

To accomplish this, the operation is the same as for finding the traces of a plane parallel to another (31). draw through the point *P* the line *PD* perpendicular to the vertical trace of the given line; this line *PD* will be parallel to the vertical trace of the plane required: its horizontal projection is *pd* (14, note; 27).

The point *d*, where this line pierces the horizontal plane, is (27) a point of the horizontal trace; which draw perpendicular to the horizontal projection of the line: and, then, through the point *A*, where this trace cuts the common intersection, draw the perpendicular *BA*, which shall be the vertical trace of the same plane.

The intersection of the perpendicular with the plane would be found as above. (35)

PROBLEM XIII.

To draw through a given point a line perpendicular to a given line.

Fig. 8. 41. Construct as above the plane perpendicular to the line; find their point of intersection; and join it with the given point: the line of junction will evidently be the perpendicular required; for, it is directed from the point to the line, in a plane perpendicular to the latter.

This solution presents itself naturally; but the following construction is both shorter and more advantageous, as it gives immediately the true length of the perpendicular.

42. Through the given point conceive a line parallel to the given line, and pass a plane through these two lines; it is evident that the perpendicular will be contained in this plane, which, therefore, being turned down upon the horizontal plane, round its horizontal trace, the whole operation may be performed in this latter.

Let *P*, *p* be the given point; draw through it *PC*, *pc*, parallel

to the given line AB, ab; find the points b and c where the two parallels pierce the horizontal plane; then will the line cbm which joins them be the horizontal trace of the plane determined by the two parallels. (28.)

In order now to lay it down on the horizontal plane, let us remark that, agreably to No. 29, the point P, p, of the plane must fall in the perpendicular dp to its horizontal trace; and at a distance $dp' = DP$, the hypothenuse of a triangle of which dp is the base and Pp' the altitude: joining CP' and drawing ba' parallel to it, the lines CP' and ba' are the the two parallels in their own plane, when turned down; therefore the line $P'Q'$ is the real length and direction of the perpendicular in that same plane.

If the projections of the perpendicular are required, they may be easily determined by restoring the plane to its former position. In order to this, produce the perpendicular $P'Q'$ until it meets at m the horizontal trace of the plane or axis of rotation: this point m will evidently remain fixed during the motion of the plane; and, consequently, is in the horizontal projection of the perpendicular, which, as it must also pass through the horizontal projection p of the given point, is therefore perfectly determined.

If the point m could not be used, the same horizontal projection pm might be found otherwise by remarking that the point Q', where the perpendicular intersects the given line, is projected in the horizontal projection ab of this same line, by a perpendicular $Q'q$ to the trace of the plane.

The vertical projection of the perpendicular would be found by means of two of the three vertical projections P, Q, M, corresponding to the horizontal projections p, q, m.

PROBLEM XIV.

To draw through a given line a plane parallel to another given line.

Fig. 9. 43. A plane is evidently parallel to a line when it passes through a parallel to it; consequently, through any point of the first line (the point where it meets the vertical plane for example) draw a line parallel to the second; find the points where both pierce the horizontal plane, these two points (33) will determine the horizontal trace of the plane required.

The vertical trace of the same is the line which unites the point

where the horizontal trace cuts the common intersection, with the point where both lines pierce the vertical plane. (22 and 33)

PROBLEM XV.

To construct the angle made by a plane and either of the planes of projections.

Fig. 10. 44. Draw a plane $A\ D\ C$ perpendicular to the horizontal trace of the given plane; this plane must be perpendicular both to the given plane and to the horizontal one, and consequently its intersections with them contain the required angle.

Construct, therefore, the intersection with the oblique plane, or, in other words, the right angled triangle whose altitude is AD and $DC = DC'$ the base: in this triangle the angle formed by the hypothenuse, and the base is the angle required.

PROBLEM XVI.

To determine the angle formed by two oblique planes.

Fig. 11. 45. Conceive a third plane perpendicular to both oblique planes, and consequently to their common intersection; this plane intersects them in two right lines, which comprehend the required angle.

As it is indifferent through what point of the intersection of the two planes the perpendicular plane be drawn, after having determined the horizontal projection ED of this intersection, let us draw any line FG perpendicularly to it. This line may be assumed as the trace of a plane perpendicular to the intersection.

But this trace FG, together with the two sides of the angle required, form in the perpendicular plane a triangle, the construction of which will solve the problem.

Conceive the plane of the triangle to revolve round its base FG; its vertex, which is in the vertical plane ED passing through the line of intersection and its projection, will remain in it during this revolution, and will consequently fall somewhere in the line ED, when the triangle comes in contact with the horizontal plane. The altitude of this triangle, which will therefore be laid off on the line ED, remains to be determined.

But this altitude, being contained in the plane *ED*, is perpendicular both to *FG* and to the intersection of the two oblique planes; since a line perpendicular to a plane is perpendicular to any line in that plane.

Let therefore the plane *ED* be turned down upon the horizontal plane round its base *ED*; and the hypothenuse *DA'* of the triangle, whose base and altitude are *ED* and *A'E*=*AE*, shall be the intersection of the two oblique planes.

But the altitude required is perpendicular to this line *DA'*; therefore it is equal in length to the perpendicular *IH'* demitted from the point *I* of the base of the triangle upon the line *DA'*.

This line *IH'*, being carried from the point *I* to the point *H*, determines the vertex *H* of the triangle *FHG*; and therefore the angle *H* formed by the two planes. The triangle *AED'* represents another way of obtaining the perpendicular *I'H*=*IH*.

PROBLEM XVII.

To find the angle made by two lines which cut each other.

Fig. 12. 46. The two lines will be known to intersect, when the point of intersection of their vertical projections, and the point of intersection of their horizontal projections, are in a same perpendicular to the common intersection (10).

Construct the two points where these lines meet the horizontal plane, and join them by a line *AB*; this line, together with the two given ones, forms a triangle, the angle at the vertex of which is the answer to the question.

In order to obtain its measure, suppose the triangle to revolve over unto the horizontal plane, round its base *AB*, and find, as in Prob. vi., at what distance its vertex will fall. Join this vertex *C'* with the extremities *A* and *B* of the base by the lines *C'B* and *C'A*. These two lines complete the triangle, and contain the angle required.

47. When two lines do not intersect, their angle is measured by that of two lines drawn through the same point and parallel to them; or, in other words, by the angle of their projections on a plane parallel to both.

Plate 2.

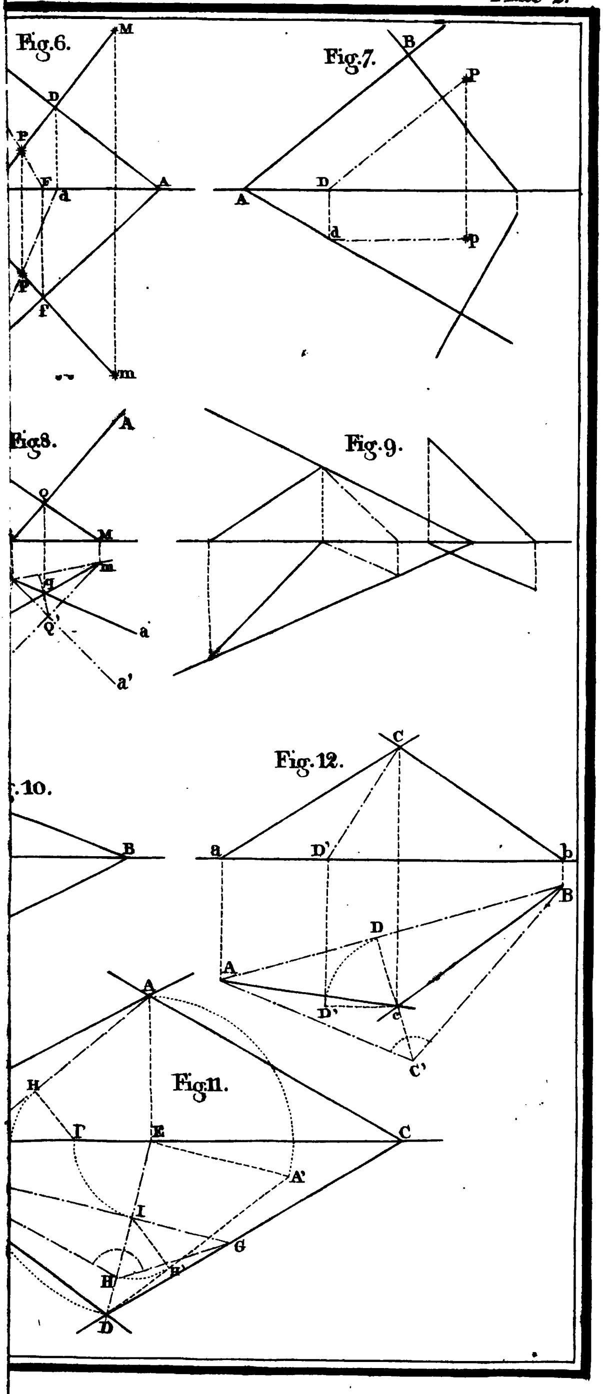


Fig.6.
M
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Fig.7.
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Fig.8.
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Fig.9.
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Fig.11.
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PROBLEM XVIII.

The projections of a line and the traces of a plane being given, to construct their angle.

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48. If from any point of the line a perpendicular be demitted upon the given plane (18), the angle made by this perpendicular and the given line will be the complement of the angle of the line and plane.

The question is, therefore, reduced to determining the angle of these two lines. (In order to facilitate the details of the operation, the figure has been made with the same letters as have been used in the preceding problem.)

Fig. 13. This angle $BC'A$, being the complement of the required angle, any perpendicular BE to BC will determine the latter.

49. The foregoing problems are all fundamental; and will be sufficient to solve all the questions that may be proposed on the right line and plane. A greater number of them would therefore increase without any advantage this elementary treatise, the object of which is only the exposition of the general principles of Descriptive Geometry. It might, however, be advantageous to solve some other problems; or, perhaps, to introduce some modifications in the data of the preceding questions, which would lead to some deviation from the general mode of solution.

As, however, a great many of these modifications will be met with in the sequel, it may be better still to proceed in this study: for, unless some doubts remain on preceding propositions, very little advantage results, in general, from exhausting a subject when it is sufficiently understood to go on in the study of propositions which depend on it: particularly, because many theories are so closely connected as to mutually explain each other; and cannot be thoroughly understood until they have been all examined.

The study of Descriptive Geometry, and especially of its applications, will be a convincing proof of the truth of this.

We will, nevertheless, conclude by a question which involves a great many of the preceding problems, and must on that account be considered, rather as a useful application, than a fundamental proposition.

PROBLEM XIX.

Two lines, which are not situated in the same plane, being given, to determine the length and position of the line that measures their shortest distance.

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50. The shortest distance between two lines is perpendicular to both.

For; conceive through one of the lines a plane parallel to the other, and project the second line upon this plane, by a plane perpendicular to it: the projection thus obtained will evidently be parallel to the line itself (Elements). Consequently all the perpendiculars demitted from the 2d. line, and which determine its projection upon the parallel plane, are equal, and at right angles, both with the projection and with the line itself. But one of them evidently passes through the point where this projection intersects the first line, and consequently is at right angles with both, since it is perpendicular to their plane.

This line, which is therefore perpendicular to the two given lines, is in length and position the shortest distance required.

For, it is equal to the shortest line which can be drawn from the second line to the plane parallel with it; and, consequently, to the given line contained in that plane.

Fig. 14th. To construct this solution, through the 1st line AB, ab, draw the plane AEb parallel to the second line CD, cd, (43): then, from any point D, d, of the second line, let fall a perpendicular DF, df, upon the plane AEb; and find its intersection H, h, with it (Prob. x); then will DH, dh, be equal to the shortest distance wanted.

This perpendicular DH, dh, together with the second line CD, cd, determines a plane perpendicular to the parallel plane AEb: its intersection with this plane AEb passes, of course, through the foot H, h of the perpendicular DH, dh; and as this intersection is known to be parallel to the second line CD, cd, it follows that the lines HI, hi, drawn through the points H, h, parallel to CD, cd, are its projections; and the points I, i, where this line HI, hi, cuts the first given line AB, ab, is the foot of the shortest distance in the plane AEb. Drawing then the lines IK, ik, parallel to the projections DH, dh, of the first perpendicular,

Fig. 13.

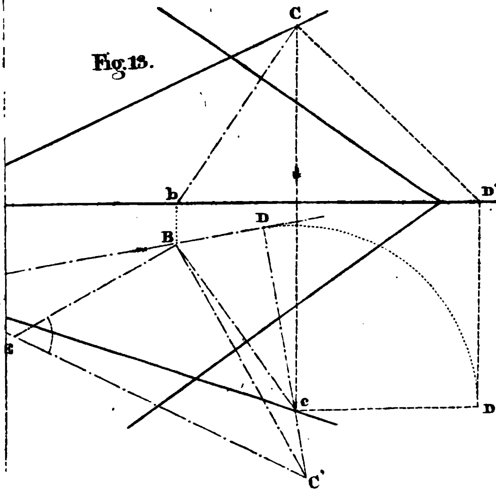


Fig. 14.

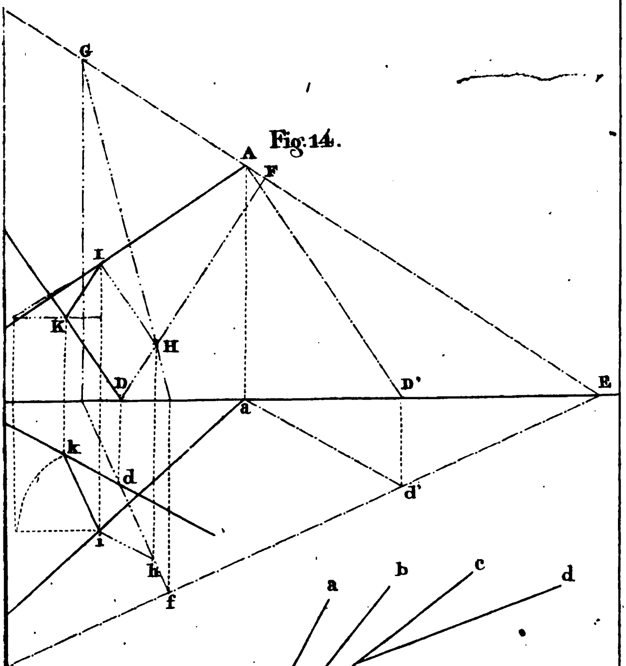
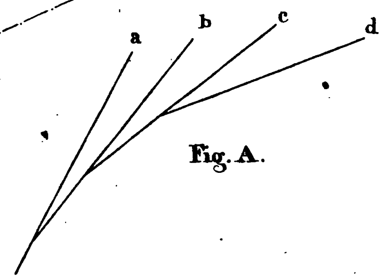


Fig. A.



the parts IK, ik, intercepted between the projections of the two given lines, are the projections of the shortest distance in its real position.

Its true length would be found as in prob. iii.

If the operation is accurately done, the points I and K must be found in the same perpendiculars to the ground line, with the points i and ~~K~~ k (10.)

CHAPTER III.

Of Curved Lines.

51. A curve has, either all its points in the same plane; and in this case it is said to be a plane curve, or else it may be such as not to coincide with any plane; it is then called a *curve of double curvature.*

52. Whether a curve be plane or of double curvature, it is projected, like right lines, by perpendiculars let fall from its different points upon the planes of projections. All these perpendiculars form a cylinder, the base of which is the projection of the curve.

The operations on curved lines, requiring therefore the consideration of cylindrical surfaces, we will speak of them in treating of surfaces; especially as, in almost every case, they result from the intersections of surfaces.

We will merely remark here that the two projections of a curve completely determine its position; since each point of it is accurately represented by its projections.

CHAPTER IV.

Of Surfaces.

53. Those Surfaces that are the object of Geometry, are all formed by certain mathematical laws, which serve to determine their parts and properties.

54. Although the number of surfaces be infinite, the principles of their generation may be reduced to a few elementary combinations.

It is, indeed, easy to understand that every surface may be considered as made up of an infinite number of curves: if, then, the law which connects one of these curves to the next is given, the surface is perfectly defined. Consequently a surface may always be conceived to be generated by a curve, which varies in its position, and often in its shape, according to some fixed mathematical law.

In order to present this in as clear a light as possible, let us explain it by some of the surfaces that are generally known.

55. 1° *Cylindrical surfaces* are known to be generated by a right line, which in all its positions remains parallel to itself, and describes a certain given curve, called the base of the cylinder, when, as in the Elements of Geometry, it is a plane curve.

It may also be described by this curve or base, moving along a right line, so as to constantly keep the same point in this line; all the other points moving in parallel lines.

In these two generations of the cylinder, the moving line changes its position, but preserves its shape.

56. The line which generates a surface by its motion is called its *generatrix*.

57. And the fixed line, along which the generatrix moves, is *the directrix of the surface.*

58. 2° *Conical Surfaces* are described by a right line which, passing always through the same point, is directed in its motion by a given curve.

Here again the generatrix varies only in its position.

But a conical surface may also be generated by a curve (which, to fix the ideas, we may suppose to be a circle) moving parallel to itself, its dimensions being every where proportional to its distance

from a fixed point called the *vertex* of the cone, while its centre, or any other point of it, slides along a line directed to the vertex.

In this mode of generation, the moving curve decreases proportionally, until it is reduced to a point at the vertex: after having passed this point, it increases again in its dimensions, in proportion to its distance from it.*

This is an instance of a generation in which the curve changes constantly, both in its form and position.

59. 3° We will take for a last example *surfaces of revolution*, which are, in general, conceived as resulting from the revolution of a curve round an axis; but may also be considered as formed by the translatory motion of a variable circle, the radius of which increases or decreases, while its centre moves in the axis of rotation; in such a manner, that its circumference may always touch the directing curve.

60. From these considerations, it follows that a surface is perfectly known when its generatrix, and the law which regulates its motion, are determined; and that, consequently, a surface cannot be represented in a more precise way than by the elements of its generation.

It would, however, be absurd to project all the elements of a surface, since it would amount to projecting every point of the surface itself; and it is very evident that such a projection could not even give an idea of the greatest number of surfaces; it would at most give the contour of those that are not infinite. But the mode of projections must be such that, at any point whatever, the generating curve may be easily determined; for, then, any required part of the surface will be immediately known.

For instance, having given the base of a cylinder and a parallel to its elements, any one of them will be determined by merely drawing a parallel to the given line from a point of the base.

In a cone the projections of the vertex and of the base will be sufficient to obtain at once any one of the generatrices of its surface; for, it must pass through the given point and the vertex.

61. But, as all surfaces are not so simple as the cone and cylinder, it is often very difficult to find, among the different methods

* It may be proper to remark here, that a cone is, like a plane, considered as indefinite. In general a surface is unlimited when it does not return into itself.

of describing them, that which will be the most convenient to obtain the results required.

Much practice and natural skill are the only guides in these graphical researches. Instead, therefore, of attempting to give some rules on this subject, which could not now appear but obscure, we will proceed immediately to the exemplification of these introductory remarks; and solve the principal problems which may be proposed on the surfaces most commonly employed in the arts. After having understood what relates to them, the general observations on surfaces, which we shall have to add, will be much more easily conceived than they would be now.

CHAPTER V.

OF TANGENT PLANES TO SURFACES.

62. The determination of tangent lines and tangent planes has, at all times, occupied the attention of mathematicians and practical men, on account of their great use both in the sciences and in the arts. In a great many instances, when it is impossible to perform operations on a curve or a surface, the lines or planes which are tangent to them are used in their stead.

63. For example, the angle of two curves is determined by that of their tangents. In like manner, the angle of two surfaces is given by the inclination of their respective tangent planes.

Tangent lines to curves, and tangent planes to surfaces, are, besides, in general, the means of forming a correct idea of them, and of determining their properties (of which the Elements of Geometry have furnished many proofs). We will consequently begin by the different questions to which tangent planes give rise.

64. *A plane is tangent to a surface, when any section made in it by a plane passing through the point of contact is tangent to the section made in the surface by the same plane.*

65. A tangent plane may also be considered as the prolongation of a small portion of the surface in every direction; for, each tangent to a section in the surface is the prolongation of an evanescent portion of the curve; consequently, since the plane contains all

the tangents, or prolongations of the sections in the surface, it may be said to be *the prolongation of the evanescent surface of contact in all directions.*

Another definition, which will be found hereafter very fertile in its consequences.

66. Inversely, all the tangents to the different sections, made in a surface through the same point, are in the same plane. But, as two lines are sufficient to determine a plane, it follows, that if two of these tangents are known the tangent plane is determined.

67. Consequently, this general problem, *to draw a tangent plane to a surface*, is reduced to the construction of two tangents to two curves on the surface; an operation which will, in general, present no difficulty, when the generatrices are of an easy description. Much, therefore, will depend on the skill and judgment with which the data of a question are selected.

68. In order to present at first the examples which require the most simple considerations, we will begin by those surfaces that contain straight lines, and consequently may be conceived as generated by them.

It is evident, that in a surface generated by a straight line, such as a cone or a cylinder, the tangent plane at a given point must pass through the right line element corresponding to this point: for, the tangent plane contains all the tangents drawn to the different sections passing through the point (62); and, because the element of the surface, being a right line, is its own tangent, it follows that it is contained in the tangent plane.

69. It is, however, a very common error, at first, to conceive the plane as tangent along the whole extent of the right line element of a surface: this is the case only in a few surfaces, which form a particular class called *Developable Surfaces*, because they may be developed so as to become planes; such are, for instance, the cylinder and the cone. As to these two surfaces, it is easily demonstrated that a tangent plane touches them at every point of the element drawn through the point of contact: for, in the cylinder, all sections parallel to the base are equal, and consequently the tangents drawn to each one of them at the point where it meets the right line element are parallel, and accordingly in the same plane.

In the cone, the parallel sections are similar; and, therefore,

their tangents are parallel as in the cylinder: whence the same conclusion.

70. But these demonstrations do not possess the generality which is required for a principle common to a family of surfaces; and, as we shall present some instances of surfaces, generated by straight lines, in which the tangent plane has only one point of contact in the element, it may not be amiss to give here a more general idea of the difference between the surfaces that are developable and those that do not possess the same property, though their generatrix be a right line.

In order to illustrate this the better, we will begin by the most simple of developable surfaces, the cylinder: conceive the base of a cylinder, a circle for instance, and a vertical plane erected through a tangent to this circle: Suppose now that the circle, being made up of an infinite number of small lines, the plane is rolled round it; the figure thus formed will be a cylinder, or prism with an infinite number of small faces, each bounded by two parallel vertical lines. Every one of these evanescent faces of the prism may be supposed to be produced every way, and will become then a tangent plane to the cylinder; since it is the prolongation of a small portion of its surface. (65)

71. Inversely, each one of the evanescent planes of contact or small faces of the prism, may be conceived to revolve round its intersection with the next face, and to be thus brought into the same plane with it. In like manner, these two faces together may turn round the third parallel side of the prism, and come in contact with the third face of it, &c.: by this successive operation, the cylinder will at last be developed in a single plane.

This formation of a cylinder, which is practised every day in rolling up a sheet of paper, will be easily extended to any other base. It will be equally easy to understand, (since each small face of the prism is a plane, which may be produced every way,) that the cylinder may be generated by the motion of a plane, which would occupy successively all the different positions represented by the evanescent faces of the prism.

72. In the same manner, a cone will be easily conceived as generated by a plane, passing constantly through the vertex, and, successively, through all the tangents to the base or directrix of the cone. The surface thus formed may be considered as a pyramid with an infinite number of small faces; and each element of con-

tact as an evanescent portion of plane, limited by two lines drawn from the vertex.

The cone is therefore developable like the cylinder; since the evanescent elements of contact with the pyramid may be progressively developed, by revolving round their successive edges.

73. The property of being developable belongs in general to those surfaces that are generated by the continuous motion of a plane. For, in all these surfaces, as in the cylinder and the cone, the successive positions of the plane will be small faces tangent to the surface. In order to explain this clearly, let us conceive any number of lines *a*, *b*, *c*, *d*, &c. fig. (A), drawn any how in a plane; then the portion of the plane included between the lines *a* and *b* being fixed, imagine that the plane turns round the line *b* to assume another position: in this second position, suppose that the portion of it, limited by the lines *b* and *c*, remains fixed; and that the rest of the plane performs part of another revolution, round the third line *c*, to occupy a third position, &c. All these different positions of the plane will form a figure, limited by the portions of it, which mark its successive situations.

But, if we suppose the motion of the plane to be continuous, as in the two above examples, the successive portions of the plane marking each position will become evanescent; the figure generated will be a surface, the elements of which being evanescent planes, will be along the whole of their extent the contact of the corresponding planes with the surface; which is, moreover, a developable surface; since the plane which generates it may retrace its successive changes.

74. Consequently, to resume: 1°. A surface generated by a straight line is developable, when its right line element may be considered as a very small plane; or, in other words, when this surface may be formed by the successive positions of a plane.

2°. In a developable surface, the tangent plane, at any point of it, coincides with the surface along the element which passes through this point.

OF TANGENT PLANES TO CONES.

PROBLEM I.

To draw a plane tangent to a cone through a point given on the surface of the cone.

75. Let us first remark that, when a point is in a given surface, its two projections must be such, that the two perpendiculars erected through them to the planes of projections may intersect on the surface.

Consequently, we cannot here assume the two projections of the given point, since they are connected by the condition of corresponding to a point of the known surface. We will, therefore, suppose that the horizontal projection of the point is given; and determine its vertical one from it.

Fig. [illegible] 76. *ee′on*, be the base of the cone (which, for more simplicity, we suppose to be a circle in the horizontal plane) and *V*, *v*, its vertex: every thing is given which is necessary to determine any part of the cone; but, in order to represent it as well

v o as possible, the tangent *vo*, *vn*, to the base, which are the extreme elements of the cone are added to the figure; as well as the two projections which include all the vertical elements of the same. Many instances will hereafter be seen of the use made, when practicable, of these limits or contours of surfaces in their representation. When they correspond to the mode of generation adopted: they generally combine the effect of a drawing, with a system of projections; and, thereby, afford much assistance in conceiving the figure and the operation.

77. Let now *p* be the horizontal projection of the given point: it is already known that the vertical projection must be in the perpendicular *pP* to the common intersection(10). But if the element of the cone, which passes through the point was known, by determining its vertical projection, we would have another line containing the same point.

In order then to construct this element, since it passes through the vertex and the point, let us connect by the line *Vp* the hori-

zontal projections of these two points, and Vp will be the horizontal projection of the element of the cone.

But the different elements of the surface pass through its base; therefore the point e, where the line vp meets the base, is in the element: In this point e, moreover, it evidently pierces the horizontal plane. The vertical projection of the point e being then E, the line VE is the corresponding projection of the element sought; and, conseqently, its intersection P with the line of projection Pp is the vertical projection of the given point of contact.

Now, according to the general method above mentioned(67), it would be required to find a line tangent to another [illegible] of the surface; but this will appear unnecessary, when we consider that the plane is tangent to the cone at any point of the right line element drawn through the vertex and the point of contact(72). It is consequently tangent at the point e, where the element meets the base; and contains the tangent eB to this base(65).

78. The tangent eB, being both in the horizontal, and in the tangent plane, is the horizontal trace of this latter.

79. The vertical trace may be determined, either by finding the point D, where the element ve, VE, pierces the vertical plane; (Prob. i.) and joining the points V & D.

Or, when the limits of the drawing do not permit it, by the following method.

Conceive through the point P, p, or any other point of the element of contact a line cp, CP, (27) parallel to the horizontal trace eB; this line is evidently in the tangent plane. Find the point C where it pierces the vertical plane; and join it with the point B common to the ground line and the horizontal trace; then will BC be the vertical trace.

80. The position of the point of a surface corresponding to a given horizontal projection, being determined by the intersection of the vertical line of projection of the point with the surface, it is easy to understand that there will often be many points of this surface for the same horizontal projection. And, consequently, the nature of the question must point out which of them to select; or else there must be as many tangent planes determined as points of contact.

In the present case, the horizontal projection Vp of the element, intersecting the base in the points e and e'; if there be no particular condition to determine whether the element passes through the

point *e* or the point *e'*, there will be two vertical projections *VE* and *VE'* for the horizontal projection *vp*; and consequently two vertical projections *P*, *P'*, corresponding in the cone to the horizontal projection *p*, through which, therefore, two tangent planes might be drawn, if the nature of the question did not specify the plane required. The construction for the second point *P'* is shown in the figure: it does not differ from the other.

PROBLEM II.

To draw a plane tangent to a cone through a point given without the surface of it.

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81. It is, first, evident that a point is sufficient to determine the tangent plane to a cone: for, as this surface may be generated by a plane (72) passing constantly through the vertex, and which is in each one of its successive positions tangent to the cone; it follows that, while it passes thus from one element to the next, one single point in its way will stop its motion.

As to the graphical construction, it is almost the same as in the preceding problem.

Fig. 15. For, since every tangent plane to a conical surface passes through its vertex, the plane required must pass through the line *MV*, *mv*, which joins the given point and the vertex of the cone: it will then be sufficient to find the point, (prob. i). Where this line pierces the horizontal plane, and to draw through it a tangent to the base of the cone, which will be the horizontal trace of the tangent plane for the same reasons as in the preceding case. the vertical trace will be found by connecting the point *B*, where the horizontal trace cuts the common intersection, with either of the points *H*, or *D*, in which the line *VM*, *vm*, and the element of contact pierce the vertical plane.

Or else, by drawing as above (76), through any point of the element or of the line *VM*, *vm*, a parallel to the horizontal trace of the plane.

The question will, when nothing particularizes it, have a number of solutions equal to the number of tangents that can be drawn to the base.

PROBLEM III.

To draw a plane tangent to a cone and parallel to a given line.

82. From the vertex of the cone draw a line parallel to the given line LN, ln; through this line the tangent plane must evidently pass.

The rest of the construction is, then, exactly the same as in the preceding problem the line VM, vm, of the figure is the parallel drawn to the given line.

VM

OF TANGENT PLANES TO CYLINDERS.

83. All that has been said of the cone answers for the cylinder, which is in fact but a cone whose vertex is at an infinite distance: consequently the lines drawn to the vertex will be, in this case, a system of parallel lines. This is the only modification the constructions of the two first problems relative to the cone will require. By comparing the 16th to the 15th figure, the solutions of the two following problems will be readily understood:

I. *To draw, through a point given in the surface of a cylinder, a plane tangent to it.*

II. *To construct the tangent plane to a cylinder, passing through a point given without the surface of it.*

The question similar to the third of the cone, would perhaps appear more difficult; we will, therefore, explain so much of it as differs from what precedes.

PROBLEM III.

To draw a plane tangent to a cylinder and parallel to a given line.

84. Since the tangent plane must pass through an element of the cylinder, (68. 71. 73). it is parallel to any line drawn parallel to the generatrix of the surface.

If, therefore, a point L, l, be assumed in the given line LN, ln; and a parallel to the elements of the cylinder drawn through this point, the line thus determined is parallel to the tangent plane; but the given line LN, ln, must also be parallel to the same; consequently the plane constructed in the figure, (43) which passes through these two lines, is parallel to the tangent plane.

But the horizontal trace of the tangent plane must be tangent to the trace or base (65, 74,) of the cylinder, and parallel to the horizontal trace of the plane just determined; therefore a tangent to the base of the cylinder drawn parallel to this trace, will be the horizontal trace of the tangent plane.

The vertical trace would be determined by drawing a parallel to the trace of the auxiliary plane, or by one of the methods given in the problems of the cone. The figure, together with the solutions of the preceding questions, will be sufficient to understand the whole of this.

85. Cones and cylinders are generally distinguished by the names of their base: they are called *circular*, *elliptical*, *parabolical*, *hyperbolical*, &c. according as the base is a *circle*, an *ellipse*, a *parabola*, a *hyperbola*, &c.

A *circular cone* is denominated *right cone*, when the *axis*, or line drawn from the vertex to the centre of the circle, is perpendicular to the base.

Or, in other words, when all its elements make the same angle with the plane of the base.

A *right cylinder* has its elements perpendicular to its base.

The *cone* and the *cylinder* are *oblique* in every other case.

OF TANGENT PLANES TO SURFACES OF REVOLUTION.

86. A surface of revolution is generated (59) by a curve revolving round a fixed axis. When the generatrix is in the plane of the axis, it is called the *meridian curve* of the surface; or, more generally, the meridian curve of a surface of revolution is the section made by a plane passing through the axis.

87. As, if this curve was not given, it would have to be deter-

mined, we will suppose here that the generatrix of the surface is a plane meridian curve: a supposition which is conformable with the generality of the cases met with, and which will be sufficient in the elementary part of this treatise.

88. The principal kinds of surfaces of revolution, employed in practical works, are the *right cone* and *cylinder*, the *sphere*, the *ellipsoid* or *spheroid*, the *paraboloid* or *conoid*, and the *hyperboloid*.

89. The axis of a surface of revolution may be conceived in any direction whatever. But, in order to introduce as much simplicity as possible in the first considerations of Descriptive Geometry, we will suppose this axis vertical. In that case the meridian curve, situated in the plane parallel to the vertical plane of projections, is projected with its true dimensions; and the circles, described by each of its points, being horizontal, will be projected upon the horizontal plane in identical circles.

Fig. 17. Through a point taken in a surface of revolution to draw a plane tangent to it.

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90. Let AB be the vertical projection of the axis; and the point a its horizontal projection. (Note No. 14).

Suppose the generating curve to be given in a plane parallel to the vertical plane: $IDD'L$ being its vertical projection. Every thing is known which is requisite to define the surface perfectly.*

Now the point C being the horizontal projection of a point of the surface, its vertical projection must be determined (75).

For this purpose, conceive through the axis and the point C a vertical plane Ca; this plane, according to the nature of surfaces of revolution, must intersect the surface under consideration in a meridian curve exactly equal to the given generatrix $IDD'L$. This meridian curve passes evidently through the point of contact, the projection c of which is given.

Through the point c, erect a perpendicular to the horizontal plane; this line being a projecting line of the point of contact will,

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* It is evident that the meridian curve is the same on each side of the axis; consequently one half only of the meridian section is necessary to represent the surface.

rizontal tangent to the circle, a line perpendicular to the meridian plane.

92. It would, perhaps, appear more methodical to proceed in the theory of tangent planes; but this first part of the treatise being elementary, the arrangement adopted in it is, without regard to systematical classification, the order which has been thought the best calculated to convey the knowledge of Descriptive Geometry with simplicity and clearness.

We will now speak of the intersections of surfaces, which it is necessary to be acquainted with, in order to understand some of the constructions of tangent planes.

CHAPTER VI.

INTERSECTIONS OF SURFACES.

93. When the generations of two surfaces are perfectly known, when for each point of them, two generatrices passing through it can be determined, then the points common to those surfaces may be exactly deduced from their nature and respective positions.

94. For, let them be both intersected by a plane; it will determine in each surface a certain curve, and it is evident that the curve of intersection with one may intersect the curve of intersection with the other surface; since they are both contained in the cutting plane. If these curves meet each other, then, evidently, the surfaces intersect in the same points as the curves.

When, on the contrary, the position of the cutting plane is such as not to meet the curve of intersection of the surfaces; then the two curves determined in them by this plane will not intersect.

95. It is easy to understand that any number of such sections may be made; and that, every section giving a certain number of points common to both surfaces, the curve connecting all these points must be their intersection.

96. The choice of the system of planes to be used for obtaining the succession of points, which form the curve of intersection, depends on the nature of the surfaces.

97. Sometimes even it is more simple to intersect them by a

system of auxiliary surfaces, as will be seen in some of the following examples. It is evident that, if two surfaces are intersected by a third one in two curves, these two curves may cut each other in many points; since they are traced on the same surface: and, in the case this takes place, the points of intersection, being in both surfaces, belong to the curve in which they penetrate each other.

98. As, in general, the intersections of surfaces are determined by a system of cutting planes, it is necessary to know, first, how to obtain the curve common to a plane and a surface. It would be easy to prove that this is always possible, when the generations of a surface may be constructed for any point of it. But these general conclusions not being necessary to comprehend the following examples, will be much better understood after them.

EXAMPLE I.

To construct the intersection of a right cylinder by a given plane.

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99. The position of the cylinder being arbitrary, we will suppose it vertical; and, as this determines only the horizontal plane, we will take the vertical plane perpendicular to the cutting plane: which is always possible.

These suppositions do not alter the generality of the solution, since the planes of projections should be changed if they were not in the above situation.

Fig. 18. Let *abc* be the base of the cylinder; and *BEF* the cutting plane.

It is evident that the extreme tangents *Aa*, *Bb*, parallel to the element of the surface, are the vertical limits of it.

If we had only to find the projections of the curve of intersection, no farther operations would be required; since the curve *abc* is the horizontal projection of the whole cylinder, and, consequently, of the intersection; and the trace *EB*, the vertical projection of the same, as well as of the plane which contains it. (23.)

100. But it may be necessary to construct the curve itself with its real dimensions. This is effected by revolving it over into one of the planes of projections; the operation will be similar to that of No. 27.

After having, by a line of projection *Cc*, determined in the trace

EB of the plane, and in the base *abc* of the cylinder, the vertical projection *C* and the horizontal one *c* of a point of the curve of intersection; if the plane *BEF* be supposed to be laid against the vertical plane by a revolution round its trace *EB*, it is evident that the point in question will fall in a perpendicular to the trace *EB*, at a distance *CC'* equal to the horizontal distance *CH* of this point from the vertical plane. (29)

For, since the plane of intersection is perpendicular to the vertical plane, the projecting line of any point of the curve is in that plane and will revolve with it: but this projecting line is besides perpendicular to the vertical trace *EB*, and accordingly equal to its own projection *CH*; whence the construction represented in the figure. Any number of points might be obtained in the same way; and would show the shape and proportions of the curve.

101. If the cylinder stood too far from the vertical plane, the curve would fall at too great a distance. In that case, the revolution of the curve might be made round any other line of its plane parallel to the vertical trace of the same; and, then, the perpendicular distances like *Cc'* should be measured from that line, or rather its horizontal projection, which, of course, must be parallel to the common intersection. This modification amounts to assuming a nearer plane of projections, or substracting equal lines from the ordinates of the curve like *CC'*. The line whose horizontal projection is *e' b'*, and which corresponds to *E' B'*, will sufficiently explain this.*

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* This way of determining the curve, by turning down its plane round a certain line is, of course, not presented here as absolutely necessary to obtain it. This could evidently be done in any position whatever, by means of its ordinates and abscisses.

In general, all the revolutions of planes, so often employed in Descriptive Geometry, must be considered as much as methods of Demonstration as of Construction. They are almost indispensable in the investigations of an Elementary Treatise, but by no means in practice: although it will always be found advantageous to perform these, as well as all other operations, in their natural position; and thereby to show the mutual dependance of the successive parts of a solution.

EXAMPLE II.

To find the intersection of a given cone by a plane.

102. The vertex and base of the cone being fixed with respect to the horizontal plane, it is always possible to assume the vertical plane, so that it may be perpendicular to the plane of intersection; for, this will be done by taking the common intersection perpendicular to the horizontal trace of it. (23)

Let *V v* be the vertex and *aaba* the base of the cone; *PM* being the vertical trace or projection of the cutting plane, and *MN* its horizontal trace.

Here, as in the preceding problem, the vertical trace of the plane is the projection of the curve; it remains only then to construct the horizontal projection of the same.

Conceive through the vertex a series of planes like *VAa* perpendicular to the vertical plane of projections; each of them will cut the cone in a certain number of elements, and the given plane in a straight line, the intersections of which with the elements will be as many points of the curve required.

In order to construct these different lines, assume any plane *VAa* in the above-mentioned situation; its horizontal trace *Aa* cuts the base of the cone in two (or more) points *a*, *a*, which each belong to one of the elements of the cone determined by the plane: therefore the lines *av*, *av*, connecting these points with the projection *v* of the vertex, are as many horizontal projections of elements of intersection. The vertical trace *AV* of the plane *VAa* is, of course, the vertical projection of all those elements. But the line of intersection of the auxiliary plane *VAa* with the cutting plane is a horizontal line perpendicular to the vertical plane; since both planes are at right angles with it; consequently the horizontal projection of that intersection will be the indefinite line *Hh*, drawn through the point *H* where the two vertical traces cut each other.

The points *h*, *h*, where this horizontal projection intersects the projection of the elements *va*, *va* previously determined, are as many points of the horizontal projection of the curve.

The construction of it, in its own plane, would be the same as in the case of a cylinder: it is indeed easy to understand that the horizontal projection is connected to the curve itself by a system of

projecting lines which form a cylinder. Consequently the curve may be considered as the intersection of its *projecting cylinder* by a plane: the operation then will not differ from the preceding one.

Erect at the point H of the vertical trace a perpendicular HH' equal to the horizontal distance hh', &c. (100)

103. It is often useful to develope a cylinder or a cone; that is to say, to spread out their surface on a plane, and with it the figures drawn thereon; so as to determine, with exactness, their parts and dimensions by the common operations of Plane Geometry. But as the shape of such figures will be altered by the developement, it is necessary to know how to find in the developed surface the position of the different points of the original one; and, reciprocally, to restore a transformed figure to its real situation on the surface. It is also sometimes required to determine the tangent at a certain point of a curve of intersection; we will, therefore, give immediately the method for performing these two operations; but it is necessary first to premise a few words on the nature of curves and of their projections.

OF CURVED LINES.

104. It has already been said that curves are perfectly determined when their projections are known.

When a curve is plane, it is seldom necessary or advantageous to perform the operations relative to it by means of projections; but, when it is a *curve of double curvature*, projections are indispensable.

The projections of a curve are connected with it by a general property, the demonstration of which has now become necessary for the ulterior investigation of the theory of surfaces, viz. that:

105. *The projections of the tangent to a curve are tangent to the projections of the curve.*

For, the perpendiculars projecting the curve form a cylinder, the base of which is the projection of it; and, as a plane is tangent to a cylinder at every point of the element of contact (94), it follows that it is also tangent at the point where the element

meets the base; consequently the tangent plane to the projecting cylinder contains both the tangent to the curve and to its projection (65) *Q. E. D.* 64

Or, else, the tangent to a curve is the prolongation of the evanescent chord of it; and, therefore, as this small chord is projected in a corresponding evanescent element of the projection of the curve, the same principle is thus demonstrated.

106. Inversely; *when the projections of a line are tangent to the projections of a curve, the line itself is tangent to the curve.*

This is a consequence of a more general property of surfaces.

107. That: *the tangent to the curve of intersection of two surfaces, is the intersection of the two planes drawn each tangent to one of them, through the point of contact in the curve.*

It is known (65) that the tangent plane, at a point of a surface, contains the tangents to all the curves which could be traced on the surface through the point of contact; consequently, if, through a point of the curve of intersection of two surfaces, a tangent plane be drawn to the first; this plane must contain the tangent to the curve: in like manner, another plane tangent at the same point to the second surface, must pass through the same tangent, which consequently is their intersection.

108. If the two surfaces are the projecting cylinders of a curve, it is evident that the two projecting planes erected through the tangents to the projections of the curve, and which are tangent each to the corresponding cylinder, must intersect in a tangent to the curve common to the cylinders; whence the preceding proposition (106) is inferred.

This being understood, let us resume the construction of the intersections of surfaces.

CONTINUATION OF THE INTERSECTIONS OF SURFACES BY PLANES; DETERMINATION OF THEIR TANGENTS & DEVELOPEMENT OF THE CYLINDER AND CONE.

Continuation of the Cylinder.

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109. Let us first construct a tangent to the intersection obtained in the cylinder No. 99. Suppose the point C' to be the point of contact given: its vertical projection will be found (100) by drawing the perpendicular CC' to the vertical trace of the cutting plane.

The horizontal projection c will be deduced from the vertical by the line of projection Cc.

Now, by No. 105, the horizontal projection of the tangent is the line ch tangent at the point c to the base of the cylinder. Its vertical projection is, of course, the trace EB of the cutting plane (23).

Having the horizontal and vertical projections of this tangent, it will be easy to find its position in the plane after its revolution: for, it will be sufficient to find any one point of the tangent, and to join it with the given point of contact.

The point h, where the tangent pierces the horizontal plane, and consequently intersects the trace EF of the cutting plane is the most convenient.

That the tangent passes through the point h, will appear obvious, by considering that both the cutting and the tangent planes contain the tangent: this line is therefore their intersection, and the point h, where their traces hc, EF, meet, a point of the same. (32)

But, since the trace EF is perpendicular to the vertical trace EB, it will after the revolution of the plane of intersection assume the position EH, upon the direction of which a distance EH, made equal to Eh, will give (100) the new position of the point h; and consequently the tangent hc', passing through it and the point of contact.

HC'

110. Any other point L, l, of the tangent might be used; and then the perpendicular LL', equal to ll', would determine the corresponding point L' of the tangent.

111. Whatever be the base of the cylinder, it is to be remarked that any subtangent, like ~~$M'H'$~~, of the curve is equal to the subtangent ~~mh'~~ of its projection or base of the cylinder. $N'L'$

l

This well known relation between the circle and the ellipse, is therefore only a particular case of a general property common to all cylinders. It exists between these two curves whenever they are intersections of the same cylinder.

Developement of a right Cylinder.

112. It is evident that the developement of a cylinder is made upon one of its tangent planes: since (71) each infinitely small element of the surface, which is also an element of the tangent plane (65), is brought successively into the plane of the first.

Fig. 18. and 18 bis. Develope, therefore, the base abc along one of its tangents,* the tangent aA for example, at any point c of the line aA (fig. 18 bis), corresponding to the point c (fig. 18) (that is to say, for which the line ac is equal to the arc ac (fig. 18)); at this point erect a perpendicular Cc equal to the altitude CH of the corresponding point of the curve in the cylinder. Then will C (fig. 18 bis) be one point of the curve in the developement required. In like manner, any quantity of such points might be determined, and the curve drawn through them with a degree of accuracy proportionate to their number.

The easiest way to rectify a circle is evidently to divide it into a certain number of equal arcs; one of each being correctly measured, the rest will be a repetition of it.

113. In this operation, as the different elements of the cylinder

* This is done by taking with a divider such a small arc of the base as may be considered as a straight line, and laying it off as many times on the line aA as it is contained in the base of the cylinder.

are made to coincide successively with the plane of the first element; a tangent to the section, being in the tangent plane passing through the point of contact (64), will, when this tangent plane coincides (71) with the plane of the developement, preserve in it its relative position to the lines of its own plane.

If it was necessary, therefore, to draw in the developement a tangent at the point C, it would suffice to make the subtangent ch equal to the same line of the fig. 18; and to join the point h to the point of contact.

Or, otherwise, to draw, from the point of contact, a line making with the ordinate Cc an angle equal to that of the tangent and the element of contact in the cylinder.

For, in the operation of opening the cylinder, neither the altitude of the point C, nor the subtangent undergo any change in their length or their relative direction.

CONSTRUCTION OF THE TANGENT TO THE SECTION OF A CONE, AND OF ITS DEVELOPEMENT.

Fig. 19. 114. In order to draw a tangent to a point h of the curve, determine the horizontal trace at of the tangent plane (77, 78): by drawing the horizontal projection vha of the element, and the tangent ta at the point a, where this element meets the base. Then will (32, 109) the point t, where the traces of the cutting and of the tangent planes intersect, be a point of the tangent: consequently the line th, which joins that point with the horizontal projection h of the point of contact, is the projection of the tangent.

115. The tangent to the curve, in its own plane, is found exactly as in No. 109: since the horizontal projection of the curve may be considered as the base of a right cylinder; and the horizontal projection of the tangent, as the trace of the tangent plane. The figure is sufficiently explanatory.

Developement of the right Cone.

Fig. 19. and 19 bis. 116. It is evident that the elements of a cone, passing all through the vertex, will, after the cone has been opened, arrange themselves round a common centre. And, because, in the case under consideration, they are all equal in length, their extremities will all be at the same distance from that centre, and consequently in the circumference of a circle.

Describe, therefore, from a point V, with a radius equal to the slant VB of the cone, an arc BAB; which make equal in developement to the base of the cone, by laying off on it the different divisions Ba, aa, &c. (fig. 19. bis), equal to the corresponding arcs ba, aa, of the base of the cone (fig. 19. bis); (these arcs are made equal to each other by the repetition of a small portion of them, as many times as it is contained in the division of the base.)

This done, the radii Va, Va, &c. (fig. 19. bis) will correspond, in the developement, to the elements of the cone passing through the points of division a, a, &c. (fig. 19)

Any point of a curve traced on the surface of the cone, will now be found in the developement on the radius passing through it; and at a distance from the centre equal to the distance of the same point in the cone, from the vertex.

Take, for example, the point H, h, (fig. 19). in order first to obtain its distance from the vertex, draw through its vertical projection H, the horizontal line HB'; and the intersection B' of this line with the extreme element of the cone will determine the real length VB' of the required distance from the vertex.

For, the horizontal line HB' is the vertical projection of a circle, all the points of which must be at an equal distance from the vertex of the cone; that is, the element projected in HV is equal to VB': Since, if it was revolved round the axis VR, it would describe the cone; and coincide, when parallel to the vertical plane, with the element $VB'B$.

This length VB', being then laid off from V to H (fig. 19 bis), on the radius VHA, corresponding to the element of the cone which contains the point, will determine this point in the developement.

Fig. 19 bis. 117. If the tangent to the curve at the point H

were required, it must be observed, as in No. 112, that, after the developement of the cone, the tangent plane at any point coincides with the plane upon which the developement has been made: and that, consequently, the triangle formed in the tangent plane, by the tangent, the element and the trace of the plane (or tangent to the base) must remain the same.

But the tangent to the base must also be tangent to the developement of this curve; since the tangent is the prolongation of a small portion of the curve, which, being contained in the tangent plane, will in every portion of it show the direction of the tangent: or, otherwise, because the tangent to the base of the cone is perpendicular to the slant of it; and, as it must remain perpendicular to it in the operation of the developement, it will be perpendicular to the radius of the circle of the fig. 19 bis; and therefrom tangent to it.

Draw, therefore, at the extremity of the radius element VHA, the tangent AT equal in length to the tangent Ta (114) fig. 19, and join TH, which will be the line required.

We have, in the two foregoing examples, supposed a right cylinder and cone: but when an oblique cylinder is given, it would sometimes be an unnecessary complication of the operation to previously determine its right base. It will, therefore, not be amiss to show the manner of operating directly in that case; as also in the very frequent occurrence of an oblique cone.

EXAMPLE III.

To find the intersection of a cylinder and a plane situated any how, with respect to the planes of projections.

Fig. (20.) 118. Intersect the cylinder and the plane ABC by a system of planes parallel to the elements of the cylinder: Each one of them will cut the cylinder in a certain number of elements, and the given plane in a straight line. The different points of intersection of this latter with the elements will belong to the section made in the cylinder by the plane.

Choose, for the auxiliary planes, a system of vertical planes: their horizontal traces, which are parallel to the projection of the generatrix of the cylinder will be the horizontal projections, both of the sections made in the surface and in the given plane.

Let *DEA* be one of the auxiliary planes : its horizontal trace, *DE*, intersecting the base of the cylinder in the points *f*, *g*, it is evident that there will be two elements of intersection, passing through these points, and projected in the line *DE*.

Their vertical projection will be determined by projecting the points *f* and *g* into the common intersection ; and drawing, through the points *F* and *G* thus obtained, the parallels *FH* and *GI* to the vertical projection of the generatrix.

But this same plane *DEA* intersects the given plane *ABC* in a straight line *AK*, *DE*, which we know how to construct (32). This line, being together with the elements *FH*, *GI*, in the auxiliary vertical plane, *DEA*, will cut them in two points *I*, *i*; and *H*, *h* of the curve of intersection.

In the same way any number of other points might be found ; and it is hardly necessary to remark that the next intersection of the auxiliary planes with the given one, being parallel to the first line obtained *AK*, their construction will be thereby simplified.

119. A tangent to the curve, being at the same time in the plane of intersection and in the tangent ~~one~~, is found, as before, by determining the trace of the tangent plane to the cylinder at the given point of contact (83) ; and then its intersection with the cutting plane (3[illegible]), which is the tangent required, (32)

fl, tangent to the base, is the trace of the tangent plane ; the point *l* is the intersection of the trace of the two planes ; and consequently *lh*, *LH*, is the tangent in its horizontal and vertical projections.

120. To find now the position and shape of the curve in its own plane, let this latter revolve round its horizontal trace *BC* ; then each point *H*, *h* of the curve revolving with the plane will fall at a distance *m H'*, which may be constructed as in No. 29 ; by making it equal to the hypothenuse *M H* of a triangle *M H h'* having *mh* for its base and *Hh'* for its altitude ; or else, as follows, by means of the different parallel intersections like *AHK*, *ED*, which have each a fixed point *K*, *D*, already known in the horizontal trace or axis of rotation of the cutting plane.

The first point *H'* being obtained as above, and, consequently too, the direction *H'D* of the line of intersection *AK*, *ED* ; as all such lines are parallel, any other point will be immediately determined by drawing, from the corresponding fixed point *D'*, a parallel to it *D' H''* whose intersection *H''* with the perpen-

dicular $h''m''$, demitted from the horizontal projection of the point upon the axis of rotation, shall belong to the new position of the curve.

121. The tangent at the point H', for instance of the revolved curve, will be constructed by joining the fixed point l of this tangent with the point of contact H'.

122. As to the developement of the cylinder, it could not be effectuated without previously determining its perpendicular base. For, in opening a cylinder, the different elements become parallel lines in the plane of the developement; and consequently, in order to fix their position, it is necessary to determine their distance; but the distance between parallel lines being a perpendicular common to them, it follows; that it will be required to have in the developement, on a right line perpendicular to the elements of the cylinder, the measure of their successive distances. But, when the cylinder is rolled up again in its former shape, this perpendicular line becomes a curve situated in a plane perpendicular to the cylinder; and which is consequently the right base of it.

This premised, the construction will be very simple; first, draw any where a plane perpendicular to the generatrix of the cylinder (40): determine afterwards the true dimensions of its curve of intersection by revolving it down as in the present problem (120); then rectify this curve; and, considering it as the base of the cylinder, lay off on the different ordinates (No. 112), erected upon the rectified base, the part of each corresponding element intercepted between the perpendicular base of the cylinder and the curve traced on its surface, and which is the object of the developement.

No particular figure is necessary to explain these different constructions; as they depend on problems already known. They may however be a useful exercise.

EXAMPLE IV.

To construct the intersection of a Conical Surface and a plane in any position whatever.

123. Intersect the cone and the given plane by a system of vertical planes passing through the vertex: the right line elements, determined by each plane in the conical surface, will be cut by the

intersection of the same plane and the given plane, one in as many points of the curve required.

Fig. 21. Since all the vertical planes pass through the vertex, their horizontal traces must diverge in all directions, from the horizontal projection a of this vertex ; so that, any line ab, drawn from it, may be assumed for the trace of one of them. Such a trace intersecting the base of the cone in a certain number of points c, c, there will be as many corresponding elements of the cone in the auxiliary plane : their vertical projections will be easily got, by projecting the points c, c, of the base upon the vertical plane, in C, C; and joining these latter with the projection V of the vertex by the lines VC, VC.

124. As to the vertical projection of the intersection of any of the auxiliary planes and the given cutting plane EFG, it might be determined in the general way : but the construction may be simplified by remarking that the different vertical planes, which pass through the vertex, must intersect each other in the vertical line, by which this vertex is projected ; and consequently the point, where this projecting line pierces the given plane, must be the meeting point of all the intersections made in it by the auxiliary planes, since this point is common to them all. After having determined then the vertical projection A of this point, by one of the known methods (27) ; the intersections with the oblique planes will be immediately obtained : for, since they are each in one of the auxiliary planes, their horizontal projections will be the same as the traces of these planes. vertical

Let then ab be the horizontal trace of an auxiliary plane ; its intersection b, with the trace of the cutting plane, will be a point of their common line (32). project the point b upon the vertical plane ; and the line BA, drawn to the converging point A, shall be the vertical projection required.

The intersection P, P of this line, with the before constructed elements CV, CV, are the vertical projections of two points of the curve ; and p, p their horizontal projections (10).

125. The position and shape of the curve in the oblique plane, will be found by ascertaining where the point A, a, must fall when the plane is turned down upon the horizontal plane (29). The point A' being this new position of the converging point of all the lines of intersection, they will immediately be constructed, by joining the different fixed points like b with it ; which will give as

many oblique lines, whose intersections, P', P', with the respective perpendiculars $P'p$, $P'p$, let fall upon the axis of rotation, from the horizontal projections p, p, shall be as many points of the curve in its true dimensions.

126. When the lines of intersection, which converge towards the point A, a, meet the trace of the plane either too far or too obliquely, some other way of determining the points must be introduced. Among the various methods, which might be resorted to, the following is particularly simple.

Suppose that the point of the curve situated in the element ad, VD, is required; draw through the point d of the base a parallel cd to the trace of the cutting plane: it is evident that a plane, which would pass through this line and the vertex, must intersect the cutting plane in a line parallel to its trace, and therefore to cd.

After having then found the point P, p, where the element ac, VC, pierces the cutting plane, it will suffice to draw through it a horizontal pr, PR parallel to dc; and this parallel will, by its intersection with the other element da, DV, give another point R, r of the curve.

127. The position of this point, when the curve is turned down upon the horizontal plane, will be found by means of a parallel $P'R'$ to cd; and which must evidently be equal in length to pr.

128. The drawing of the tangent is exactly as in other cases.

The figure, together with the preceding problem referred to, is sufficient to completely understand the construction.

129. As to the development of the curve of intersection, it requires the previous solution of another problem which must come in order. (170.)

130. It may be proper to remark, that the oblique plane may cut all the elements of a cone; in that case the curve will return into itself. It is called *ellipse* in the right cone.

131. Or, else, it may be parallel to two elements: as, in this supposition, the two elements can never be intersected by the plane, the curve will be infinite; and have two distinct branches on each side of the vertex. But, since any other element, however near it may be to the parallel ones, will be intersected by the plane; it follows that the curve will approach nearer and nearer to these parallel elements, which for this reason are called the *asymptotes* of the curve.

In the cone of the second *order** this curve takes the name of *hyperbola.*

132. Lastly; the cutting plane may be parallel to a tangent plane; and then it will intersect all the elements but that of contact; the curve, therefore, will be infinite, it will have but one branch and no asymptote. In the particular cone already mentioned this curve is called *parabola.*

It has not been deemed necessary to add any figures to these remarks, because such figures will be easily made: and, besides, the curves mentioned are so universally known, that the general conic sections, of which they are particular cases, will be readily conceived.

We will, besides, resume what relates to them in the additions.

EXAMPLE V.

To find the intersection of a surface of revolution by a ~~giving~~ given plane.

133. Let us suppose that the axis of the surface is vertical; and the cutting plane perpendicular to the vertical plane of projections.

The curve *ABAB* parallel to the vertical plane, being the meridian section or generatrix of the surface (87). If we conceive a system of horizontal planes to intersect the surface: each one will determine in it a circle; and in the given plane a horizontal line, the intersections of which, with the circle, will be two points of the curve.

Fig. 22. *BB* being one of the horizontal sections; the centre of the circle is the point *o*, and its radius the ordinate *oB* of the meridian curve. This circle is, of course, projected in the horizontal plane in its identical shape. If, therefore, from the point *a*, with a radius equal to *oB*, a circle be described, it will be the projection of the horizontal section *BB*. But the intersection of the same horizontal plane *BB*, and of the given plane is the indefinite horizontal line *Cc*, which will determine in the circle just described the two points *c*, *c* of the horizontal projection of the curve.

* That is, which has a conic section for its base.

134. In order to draw a tangent at a given point c, C, of the section, it must be supposed that we know how to draw a tangent to the generatrix of the surface: the tangent plane will then be easily constructed, and therefrom the tangent required.

To determine the tangent plane, draw a tangent BD at the point B of the generatrix; this tangent pierces the horizontal plane at the point D, d': but this tangent being made to revolve, together with the generatrix, round the axis AA, will, in all its positions, be tangent to the corresponding meridian curve. It will consequently be tangent at the point C, when the point B coincides with it. In that position it will, of course, pierce the horizontal plane in the horizontal trace ca of the meridian plane, passing through the point C, c; and at a distance da from the axis equal to DA; since the point D, where the tangent BD pierces the horizontal plane, will revolve with it round the axis.

Through the point d, draw, therefore, a perpendicular df to the radius da; it shall be the trace of the tangent plane (90). And consequently fc, the horizontal projection of the tangent (114).

125. It is evident that the points F, F are the two *vertices** of the curve of intersection.

136. The curve and its tangent, in the cutting plane, would be found by a revolution of this plane exactly as in the cylinder (99).

137. It has been supposed here, that the meridian curve was given: but sometimes the generatrix differs from it. However, it is always easy to deduce the meridian section from the generatrix: for, each point of this latter describes round the axis a circle, the intersections of which, by the meridian plane, are two points of the meridian curve.

The following remarkable example will illustrate this; and show, at the same time, that it may often be advantageous to use a generatrix different from the meridian curve.

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* That is: the extremities of a line generally called *diameter*, which divides the curve symetrically.

GENERATION OF THE HYPERBOLOID OF REVOLUTION.

138. The surface meant here results from the revolution of a hyperbola round its conjugate axis. But this mode of generation is not so convenient for graphical operations as the following.

139. Conceive a straight line to revolve round another, at a certain distance from it; the generatrix preserving, in every position, the same relative situation, with respect to the axis of motion.— The surface thus generated will be a hyperboloid.

Fig. 23. The figure shows the generation of this surface: Every point of the revolving right line describes a circle, of which the centre is in the fixed line or axis. The large circle, in the figure, is that traced on the horizontal plane, by the point of the generatrix situated in it: the small circle is the horizontal projection of the circle described by the horizonal line, which measures the shortest distance between the axis and the generatrix (50). The horizontal projection of any element of the surface is evidently tangent to that small circle: for, the generatrix cannot approach nearer to the axis than the length of its radius.

The horizontal projection of the element parallel with the vertical plane is then the line *ab*, tangent to the small circle *mon*; its vertical projection *AB* may be given; and then the height of the small circle will be determined by the intersection *N* of the generatrix *AB* and the axis.

Or else the vertical projection *MNM* of the small circle being known, the projection of the element will be found, by uniting the point *N* with the point *A*.

140. Any other element *cd* would be found, by drawing to the small circle the tangent *cd*; from which the vertical projection would be deduced, by projecting the point of contact *m* in the vertical projection *MM* of the small circle, and the point *c*, where this element pierces the horizontal plane, in the point *C* of the ground line.

141. The regularity of the figure shows, that there will be above the small circle, and at an equal distance from it, a circle *BB* equal to the base *AA* of the hyperboloid: any element *cd*, *CD*, might, therefore, be obtained, without having to determine a point of con-

tact, by projecting as before the point c in C and the point d, considered as belonging to the upper equal circle, in the point D of the line BB.

142. If a cylinder be imagined to have the same axis as the hyperboloid, and the small circle MNM, mno for its base; it is obvious that the generatrix of the hyperboloid must be tangent, in its successive positions, to this vertical cylinder, and make a constant angle with its elements. This is another generation of the surface under consideration.

But it is evident, that, in the tangent plane to this central cylinder, we can conceive on one side of the element the generatrix of the hyperboloid; and on the other, in an inverted direction, a line making, with this same element of the cylinder, an angle equal to that of the generatrix with it. This new line and the generatrix, being symetrically situated with respect to the cylinder, will have their corresponding points equally distant from the axis of rotation; and, consequently, whether one or the other revolves round the axis, the same circles shall be described and the same surface generated.

143. The same hyperboloid being described indifferently by either of the two lines, symetrically situated on each side of the axis; it follows, that, through the same point of the surface, two symetrical elements can always be drawn in two opposite directions; whence the following very simple method of drawing a tangent plane at a given point of the hyperboloid.

PROBLEM.

To draw through a given point of the surface of a hyperboloid a plane tangent to it.

144. The horizontal projection m of the point being given, draw through it the two tangents bma, bma to the inner circle; these horizontal projections of the elements which pass through the given point, intersect the large circle or base of the hyperboloid in two points b and b, where the elements evidently pierce the horizontal plane (139). The line bb, which passes through them, is, therefore, the horizontal trace of the tangent plane: for, it is well known that it must contain both elements (68).

145. The vertical projections of these elements would be found,

by projecting the points b, b, in the lower circle CD, and the points a, a, in the upper one $CAAD$: the intersection M of the two vertical projections AB, AB, is the corresponding projection of the point of contact; and must, therefore, be contained in the line of projection Mm.

As to the vertical trace of the plane, it would result from the determination of the points F, F, where the two elements pierce the vertical plane.

146. It is evident here, that the plane is not tangent along either of the elements of contact: since its trace bb is a chord of the base of the surface; and would be a chord to any other circle assumed as the base of the same; unless it were that passing through the point of intersection of the two elements: consequently the plane is tangent only at that point, and the surface is not developable. (74).

147. Another remarkable property of this surface is: *that any plane, drawn through one of its right line elements like* ba, BA, *is tangent to the surface, somewhere in one point of the element*: for let bb be the trace of such a plane: if, through the other point b, where this trace cuts the base of the hyperboloid, another element ba be drawn; it must, since it is in the surface, intersect the first element. But the point of intersection m is in the assumed plane, because it is in the generatrix, through which the plane passes; consequently, this plane, containing the points m and b of the second element, must also contain this latter; and is, therefore, tangent at the point m, since it passes through two elements, crossing each other in that point.

THEOREM.

The surface generated by a straight line revolving round another, and preserving always the same distance and angle with it, is a hyperboloid.

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148. As, for the sake of conciseness, the name of the surface has been used in the preceding question, it may not be unnecessary to prove that it is a hyperboloid; although such a demonstration belongs rather to the application of algebra to geometry.

It is first obvious, that the meridian curve is the same as the contour of the vertical projection of the surface: and that this

contour is determined by the successive intersections of the elements of the surface (Fig. 23 shows it plainly). The vertical projection of each element is consequently tangent to this contour.

149. If a more minute demonstration were required, it might be observed: that the contour is formed by a cylinder perpendicular to the vertical plane, and tangent to the surface; consequently a plane tangent to the cylinder must also touch this surface. But it is known, that the tangent plane must pass through a right line element of it: this plane, besides, being perpendicular to the vertical plane, has for its base the projection of the element; which must be tangent to the base of the cylinder, (83, 78), or projection of the contour.

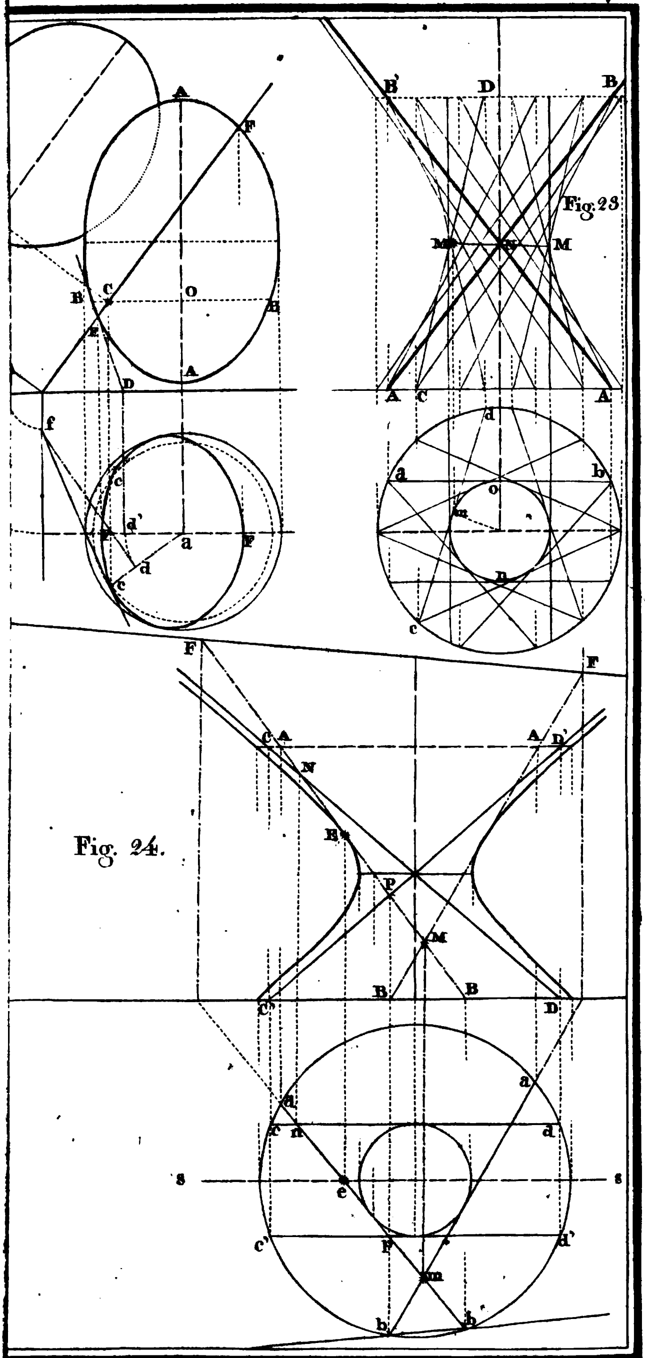
150. That the contour is the same as the meridian curve, may be as easily demonstrated. For, the cylinder, which determines it, is composed of horizontal elements, which, since they are tangent to the surface of revolution, must each touch one of the horizontal circles of the same. But those elements of the cylinder being all parallel, the perpendicular radii, drawn from the centres of the different circles to their point of contact, are parallel; and in a plane perpendicular to the cylinder. Or, in other words, the meridian plane, drawn through the axis, and perpendicular to the elements of the cylinder, contains all the points of contact of those elements with the surface, and, consequently, the contour.

150. After having noticed these properties of the surface in question, let us remark: *that the contour or meridian curve will result from the intersections of all the elements with the meridian plane ss.*

152. Project now upon the vertical plane in CD and $C'D'$ the elements cd and $c'd'$ parallel to it (139): any other element, like ab, AB, intersects the element cd, CD in a point n, N, and the line $c'd'$, $C'D'$ in the point P, p. But the point e, E (151), belongs to the meridian curve; and, as ne is equal to ep, it follows that their vertical projections NE, EP, are also equal; hence the curve has the property, that a line PN, drawn between two fixed lines CD, $C'D'$, is tangent to it (149), and divided equally by the point of contact.

The meridian curve is, consequently, a hyperbola; and the surface generated a hyperboloid.

153. To complete the demonstration, let us remark that the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F
d'
m
b
b'

Fig. 23

Fig. 24.

two generatrices cd, CD; $c'd'$, $C'D'$, parallel to the meridian plane, will not meet this plane; and that, consequently, the points of the curve, situated in them, are at an infinite distance: whence it must be concluded, that they are asymptotes to the curve; as would also be deduced from the above property of the meridian curve (152).

EXAMPLE VI.

To construct the curve resulting from the intersection of a hyperboloid by a plane.

Fig. 25. 154. We will suppose that the plane is perpendicular to the vertical plane of projections.

If we conceive a system of horizontal planes, intersecting both the hyperboloid and the plane, they will give, in the surface, circles; and, in the plane, horizontal lines, the intersections of which will determine the curve required (94).

155. This construction does not necessitate the previous determination of the meridian curve: for, each circle has its centre in the axis, and a point in the generatrix; this point, once obtained, the radius of the circle is known, without having recourse to the meridian curve.

156. Let ab, AB, be the projections of the generatrix, when parallel to the vertical plane; and CDE the cutting plane. Then, any horizontal plane HH intersects the element AB in a point G, g, of a circle, the radius of which is og; and which cuts the horizontal line Fif, determined in the plane CDE by the auxiliary one HH, in the points i, i, which belong to the horizontal projection of the curve.

157. It may be advantageous to construct separately the vertices of the curve; as they could not be got directly by the general construction. The following method may be used.

It is, first, evident that these vertices must be in the meridian plane OE perpendicular to the plane of the curve, since it divides the operations symetrically.

This meridian plane intersects the plane CDE in a line, the vertical projection of which is DC; that line contains the two vertices. If it now revolves round the axis of the hyperboloid; it will generate a cone, and the two vertices, two circles which

must be, both in the hyperboloid and in the cone, of which they are consequently the common intersection; it remains then to find these circles.

But they evidently have, as well as all other circles of the surface, a point in the generatrix ab, AB; which point must then be the intersection of the cone and of the element ab, AB; the problem is therefore reduced to: *finding the intersection of a right line and a cone.*

158. The point V, where the revolving line DC meets the axis, is the vertex of the cone; and the circle Efl its base.

Draw through the vertex V, o, a line VM, om, parallel to the given generatrix ab, AB; and through these parallels pass a plane, the trace of which will be lb, ml (43). This trace intersects the base of the cone in two points l and l; and, consequently, ol, ol are the horizontal projections of the elements, cut out of the cone by the plane. These elements meet the generatrix ab, AB, in the two points of intersection required n, n.

Describing then, from the point o, and through these points, the two circles ns, ns, they will contain the vertices sought, which will be, in the horizontal plane, the intersections s, s, of the circles with the horizontal projection Eo of the axis of the curve.

159. Any other point of the curve might be determined, in a similar manner, by a cone described by the line of intersection of the meridian plane passing through the point and the given plane. This construction is advantageous near the vertex of the curve, where the intersections of the circles and the corresponding horizontal lines become too oblique.

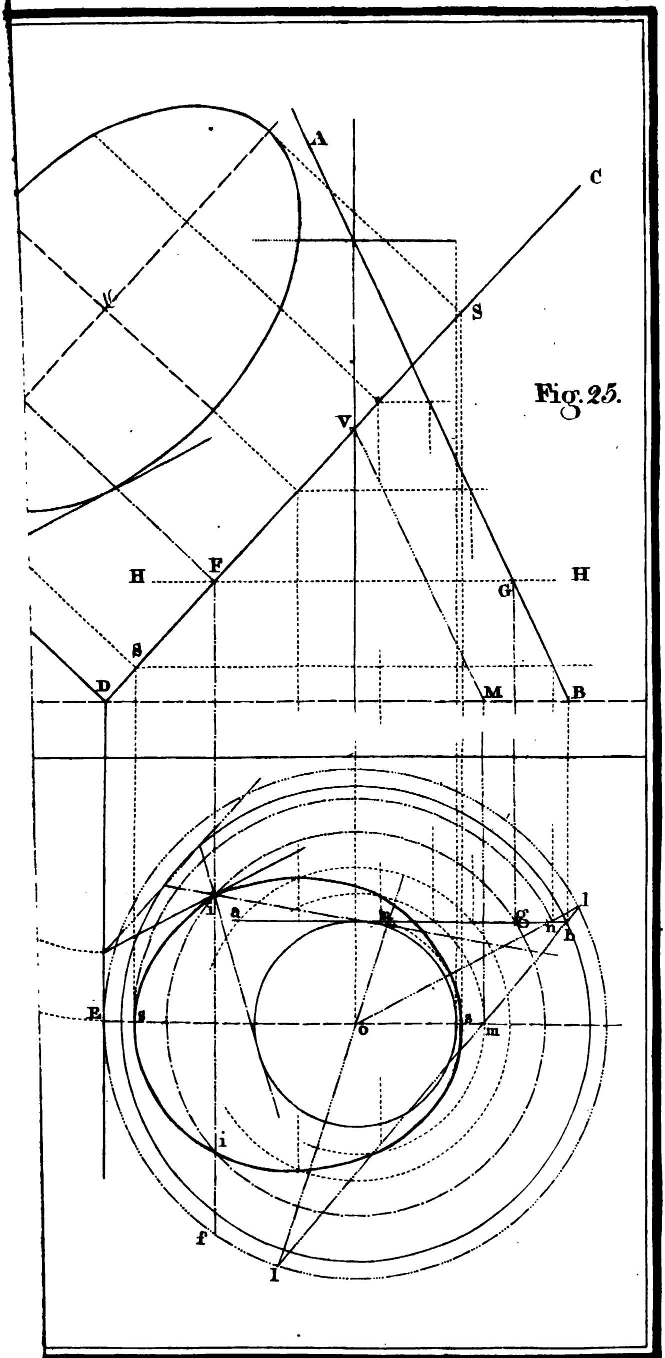
160. A tangent to the projection of the curve will be drawn, by determining the tangent plane corresponding to the point of contact (144); and then its intersection with the cutting plane (114), &c.

The curve and its tangent in its own plane will be found as in No. 99: for these two constructions, see the figure.

161. The curve of intersection will be a *parabola*, an *ellipse*, or a *hyperbola*, according as the angle, which the intersecting plane makes with the horizontal plane, is equal to, less, or greater than, the angle of the generatrix.

The demonstration of this will be found in the second part, together with some additional remarks. It may, however, be advantageous to construct graphically these three different cases;

Plate 8.



for, it is only by a great number of applications that a perfect knowledge of the principles can be obtained.

The foregoing examples of intersections of surfaces by planes are nevertheless sufficient to illustrate the general method of obtaining them.

EXAMPLES OF INTERSECTIONS OF SURFACES BY OTHER SURFACES.

EXAMPLE I.

To construct the intersection of two cylindrical surfaces.

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162. Conceive a system of cutting planes parallel at the same time to the elements of both cylinders; each of these planes will intersect the first cylinder in a certain number of straight line elements; and likewise the second cylinder in some of its elements: the lines, thus determined by the same auxiliary plane in one surface, will intersect those of the other in a number of points, which will all be in both cylinders, and consequently in their curve of intersection.

Let *hpqa*, *brsd*, be the traces of the two given cylinders; *LM*, *io*, a parallel to the elements of the first, and *LN*, *zn*, a parallel to the elements of the second.

Through the second line *LN*, *zn*, draw a plane parallel to the first; by assuming in the line *LN*, *zn*, a point *L*, *l*; drawing through it a parallel *LM*, *lm*, to the line *LM*, *io*; and then joining the points *m* and *n*, where the lines *LM*, *lm*, and *LN*, *ln*, pierce the horizontal plane, by the trace *mn* of the parallel plane required (43).

The plane whose trace is *mn*, being parallel to the elements of both cylinders, any other plane parallel to it will, of course, be in the same situation with respect to them; and, as the traces of all such planes must be parallel, any line *pqrs* parallel to *mn* may be assumed as the trace of one of the auxiliary planes.

This trace *pqrs* cuts the base of the first cylinder in the points *p*, *q*, through which we can draw immediately the horizontal pro-

jections pt, qx, of two elements, without constructing the vertical trace of the plane; which is unnecessary, since we know the foot of the elements and their direction.

In like manner, through the points r, s, where the same trace intersects the base of the second cylinder, draw the two elements ru, st, of this latter; they will intersect the two others in the points t, u, v, x of the horizontal projection of the curve.

The vertical projections of the same will be found at the intersections of the vertical projections PT, QX, of the elements of one cylinder, with the projections RU, ST, of the others; and the correctness of the operation will be verified, if the points U, T, V, X are found in the same lines of projection with the corresponding points u, t, v, x.

163. The curves of intersection of surfaces are, in general, of double curvature; and it is only by a great number of points that their shape may be ascertained. For this reason, it is very advantageous to begin by the most characteristic points of the curve: among others, its limits in different directions are particularly useful: such are the points situated in the extreme elements of each cylinder, in which they are tangent to the curve.

Let us take, for example, the extreme element hj: if a trace $h5$ be drawn through the foot h of this element, it will cut the base of the other cylinder in the points 4 and 5, through which the elements 4,6; 5,7 being drawn, will give the two points of contact 6, 7, of the extreme element hj and the curve. Any other such points, either in the vertical or in the horizontal projection, would be found in the same way.

A plane like 1, 2, tangent to one of the cylinders, and intersecting the other, must also give a certain number of elements tangent to the curve, together with their points of contact; since this plane is one of the limits of the curve.

It would, however, be useless to enter into the discussion of all these details of construction, which practice alone can teach; the object of the few observations made here, is only to point out the possibility of performing the operation with more accuracy and simplicity. The figure has been so combined as to obtain the curve with as few lines as possible, and will explain the manner of doing it better than words.

164. One of the cylinders often penetrates the other; there are then two curves of intersection: this will be the case, when

(as in the figure) the two extreme tangent traces to one of the bases intersect the other; for, then, the intersection will be discontinued, from one side of the cylinder to the other.

165. The tangent to the point *u*, for example, of the curve will be determined, by drawing the horizontal trace *pk* of the plane tangent at the point *u* to the first cylinder; and the trace *rk* of the tangent plane at the same point in the second cylinder: the line *Ku* drawn from the point of intersection of these two traces to the point *u* of contact is the horizontal projection of the tangent (107, 83, 32).

The point *k* being projected at *K* in the common intersection, the line *KU* shall be the vertical projection of the same tangent.

166. If, for any purpose whatever, it was necessary to fix together two hollow cylinders, so that they might perfectly fit each other; it would be required to have, in each, their common curve of penetration. This would be accomplished by developing them separately; and tracing the curve on each developement. In this way, the two cylinders, being rolled up again, must perfectly join each other at their intersection. The operation for obtaining these developements would be: to find, as in No. 122, the right base of each, and to operate in every respect as in the same article.

EXAMPLE II.

To construct the intersection of two conical surfaces.

167. If a series of planes are made to pass through the two vertices, they will cut each cone in a certain number of elements; the intersections of which, with those of the other, will be as many points of the curve of penetration.

The horizontal traces of these auxiliary planes will all pass through the point *a*, where the line *VS*, *vs*, which joins the vertices, pierces through the horizontal plane.

Any line, *ab*, passing through it, may, therefore, be assumed as the trace of one of the planes; its points of intersection with the bases of the cones, being united to their respective vertex, the horizontal projections of as many elements of both cones will be determined. Their intersections *t*, *u*, *v*, *x*, in the horizontal and *T*, *U*, *V*, *X*, in the vertical plane, are in the curve.

The construction is so much like that for cylinders, that this short explanation, with the help of the figure, must suffice to understand the whole of it.

168. The tangent at one point of the curve will also be constructed, as in the preceding example, by drawing the tangents to the base of each cone; finding their intersection, &c. (fig. 26.)

169. It is of some importance to distinguish by dots those parts of the curve of intersection that are, as it were, hidden by either of the surfaces. The different points, that belong to these masked parts, result in general from the intersection of two unseen elements. It requires but very little attention to mark these differences: they have been carefully established in the figures 25 and 26.

The delineation of the shape of the curve on the developement of one of the cones requires the solution of the following problem.

EXAMPLE III.

To construct the intersection of a conical surface and of a sphere.

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170. We will suppose that the vertex of the cone is in the centre of the sphere, because it is the particular case wanted here.

Any number of planes, passing through the common centre of the sphere and cone, will each furnish in the sphere a great circle, and in the cone a certain number of elements, the intersections of which, with the great circle, shall be as many points of the curve.

As the choice of the planes is indifferent, we will take them vertical.

Fig. 27. The point v, being the horizontal projection of the vertex, any line va drawn through it may be assumed as the trace of one of the planes, and, consequently, as the horizontal projection of the elements of the cone, and of the great circle of the sphere: the points a, a, where this trace cuts the base of the cone, being projected upon the vertical plane (120), the vertical projections of the elements will be VA, VA.

As to the great circle of the sphere, it would be projected in an ellipse; to avoid the description of which, we will cause the vertical secant plane to revolve round the projecting line Vv of the centre. By this means, the circle may be brought in parallelism with the vertical plane, and will, then, coincide with the great

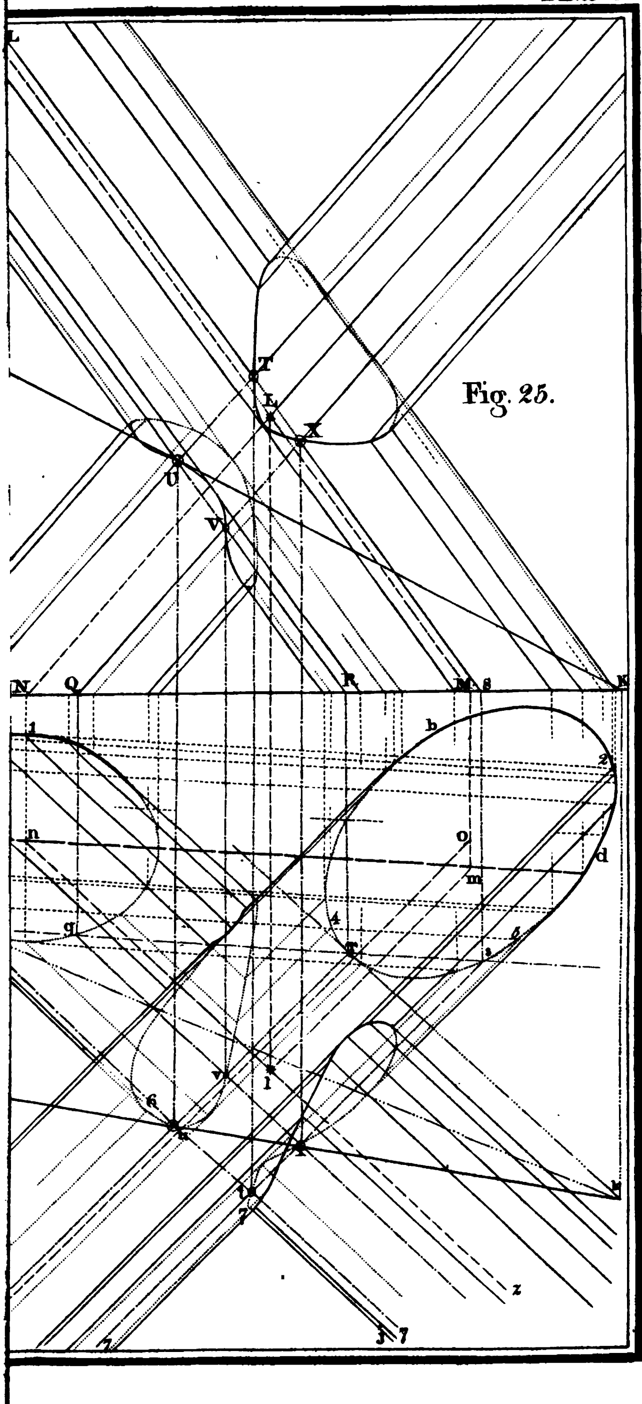

Fig. 25.

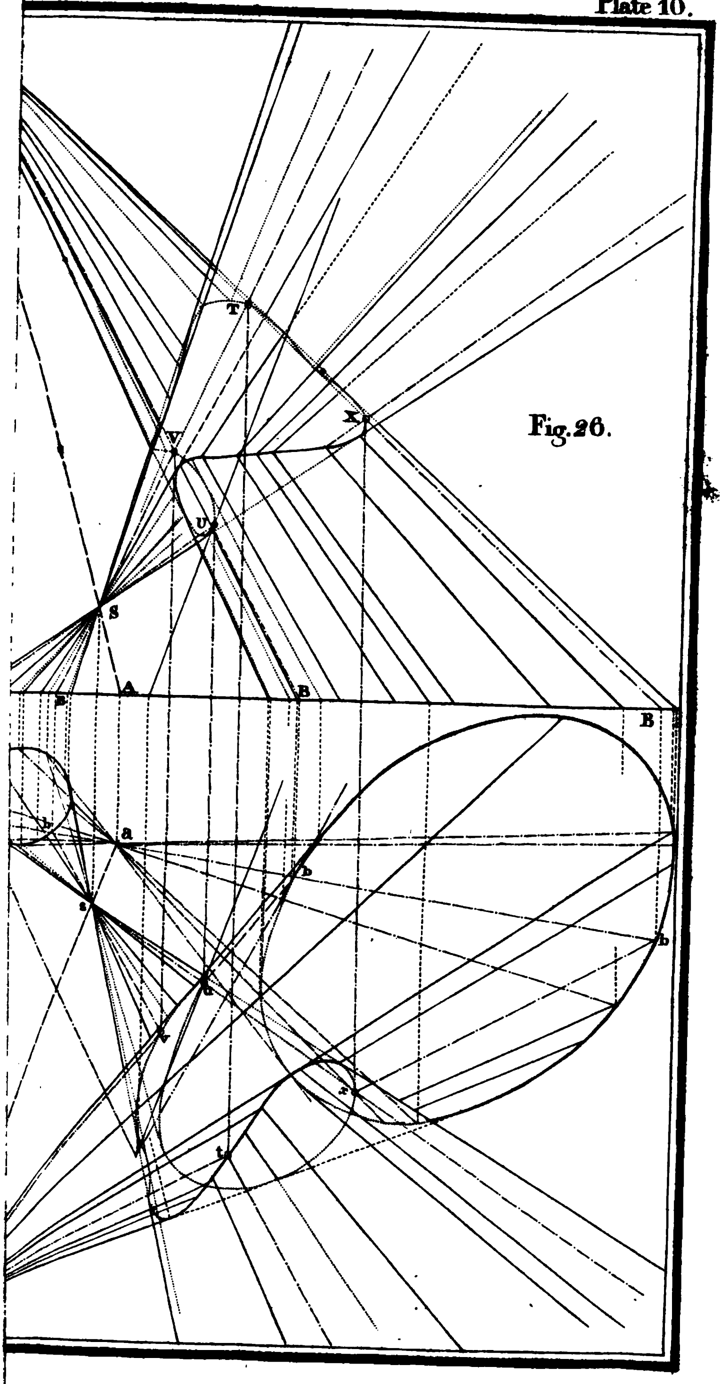

Fig. 26.

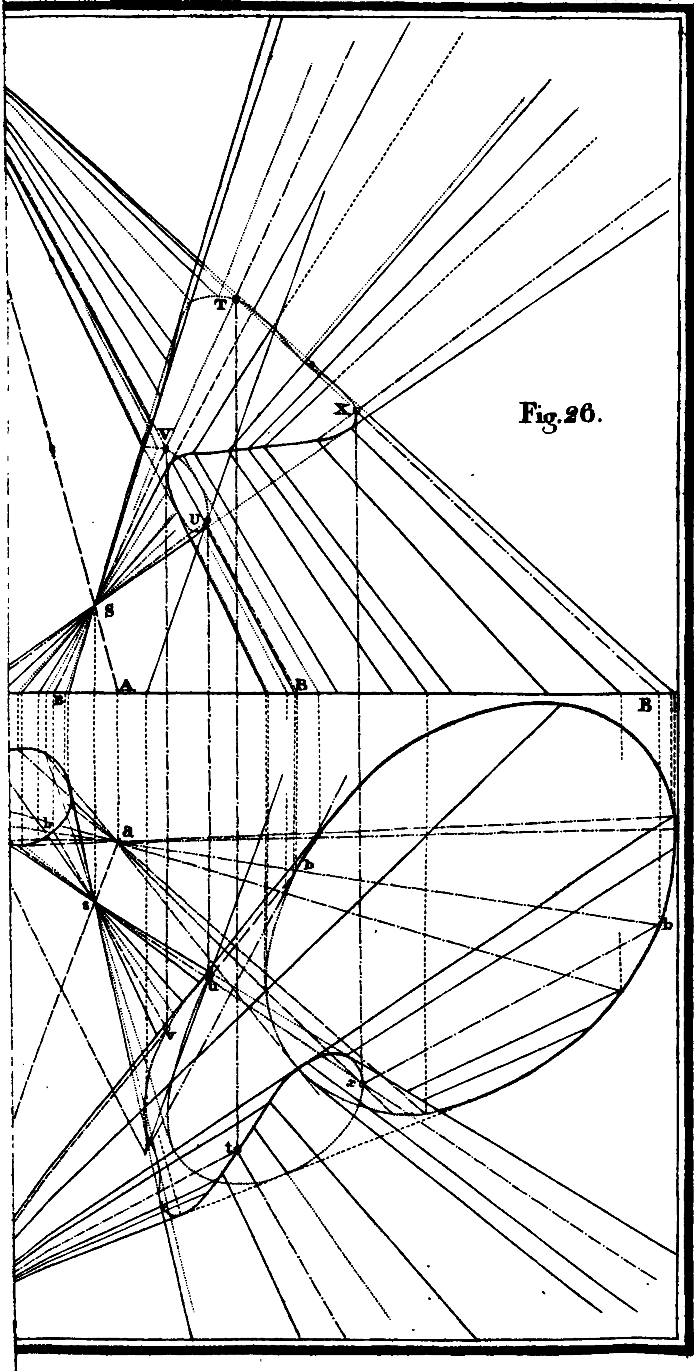

Fig. 26.

circle projected in *OB'O*. The two elements of the cone, being carried round by the same plane, the points *a*, *a*, will describe, in the horizontal plane, the two arcs *aa'*, *aa'*, round the centre *v*; and coincide with the points *a'*, *a'*, when their plane becomes parallel to the vertical plane of projections.

The vertical projections of the elements of the cone will then be *VA'*, *VA'*; by which the great circle is intersected in *B'*, *B'*.

These points of intersection, when the auxiliary plane returns to its former position, will describe horizontal circles, the vertical projections of which are the horizontal lines *B'B*, *B'B*. But the same points are also contained in the projections *VA*, *VA* of the elements of the cone; and will consequently be at their intersections *B*, *B*, with the lines *BB'*, *BB'*.

The curve, drawn through all the points determined like *B*, *B*, is the vertical projection of the intersection required.

The points *B*, *B*, being projected down in the trace *ava* of the auxiliary vertical plane, will give two points *b*, *b*, of the horizontal projection of the same curve.

171. The tangent at any point *b*, *B*, of the curve would be found by drawing a tangent *af* at the point *a* of the base of the cone; this line, being the trace of the plane tangent to the cone in the point *B*, *b* (78), its intersection *f* with the horizontal trace *ef* of the tangent plane (104) to the same point in the sphere, must be a point of the tangent, the horizontal projection of which is, therefore the line *fb*.

The vertical projection of the same is evidently *FB*.

The fig. shows the construction of the tangent plane to the sphere, which is the same as in No. (90). *B'D* perpendicular to the radius *B'V* is tangent to the meridian circle; it pierces the horizontal plane in the point *D*, *d*, which being brought to its natural position, by a revolution round the centre *v*, comes in *e*. The perpendicular *ef* to the line *aa* is then the trace of the tangent plane.

172. If the sphere and the cone were not concentric, the operation would be nearly the same; a series of cutting planes must then be made to pass through the line, joining the vertex of the cone and the centre of the sphere; and each one be revolved to a parallel position with respect to one of the planes of projections.

This last construction might be simplified, by taking the vertical plane parallel to the line which connects the two centres, round

which the revolutions of the different planes would be easily performed.

PROBLEM.

To construct the developement of a cone; and to trace on this developement, the shape of a section of the cone.

173. When a conical surface is unfolded on a plane, its elements arrange themselves round a common centre, corresponding to the vertex of the cone; and the different points of the surface retain their respective distances from the vertex. The position of any point will consequently be known, on the developement, provided that the situation of the element, which contains it, may be determined.

174. The respective positions of a certain number of lines, converging towards a common centre, are generally established by means of the angles they make with each other: but the measure of an angle is the arc described from its vertex; therefore, in order to fix the position of the elements of the cone, it is first requisite to find on its surface such a curve as will become a circumference of a circle in the developement. This condition implies that the points of this curve, must all be at the same distance from the centre or vertex of the cone.

The curve required shall therefore be situated on the surface of a sphere, described from the vertex of the cone, as a centre, with an assumed radius.

175. After having then constructed, as in the preceding problem, the projections of the intersection of such a sphere, with the cone, describe from the point V (fig. 29), considered as the vertex of the cone in the developement, and with a radius VM equal to that of the sphere, a circumference MBM; it is evident that the different points of the curve of intersection will, after the developement of the cone, be found in this circumference.

But in order to lay off, on the circumference, the different parts of the curve, their length must be first obtained; this will require a double developement.

For, the curve being in general of double curvature cannot, as in the cases above examined of plane curves, be turned over on the plane of projections, to measure the real length of its different

arcs and rectify them.* It must, therefore, undergo a double operation: the first to make it a plane curve; the second to convert it into a circular arc.

176. To this end, let us remark that the curve and its horizontal projection belong both to a vertical cylinder; this we can immediately develope since the right base of it is known (119). This right base or horizontal projection *blbm* being rectified, along the line *mbm* (fig. 28), the curve itself *MBM* will be obtained, as in No. 112, by means of the successive abscisses *mb*, *bc*, *cc*, *cc*, &c. and their corresponding ordinates *mM*, *bB*, *Cc*, *cC*, &c. (fig. 27 and 28).

177. The tangent at a point *B* of the new curve shall be the line *Bf*, such that *bf* (fig. 28). may be equal to *bf* (fig. 27). (113).

178. Now the different portions *MB*, *BC*, *CC*, &c. of the curve, can be measured and laid on the circumference *MBM* (fig. 29), previously described (175), from *M* to *B*, from *B* to *C*, from *C* to *C*, &c.

The whole circular arc *MBCBM* will, when the developement is completed, be the length of the curve; and the whole area *MBMV* limited by the radii *MV*, *MV*, will be equal to the surface of the cone, between the curve and the vertex.

179. The point *B* (fig. 29). being in the developement the position of the point *B*, *b* (fig. 27) of the surface, the radius *VBA* corresponds to the element *av*, *VA* (fig. 27); and any point *A* of it will be found, by laying off from the vertex or centre *V* (fig. 29). the line *Va* equal to the distance *VA'* of the corresponding point *a*, *A*, of the surface, from its vertex (fig. 27).

The curve *NAN* (fig. 29) represents on the developement the base of the cone of the figure 27.

180. The tangent to the curve, at the point *A*, would be constructed, by drawing the tangent *Bf* to the circle *MBM*, taking *Bf* equal to *Bf* (fig. 28); and joining *fA*. For, the triangle *ABf* cannot be altered by the developement; and it is evident, that the radius *BV* of the sphere must be perpendicular to the tangent *Bf*, as well as to any other line of the tangent plane: consequently, the triangle *ABf* has a right angle at *B*, which, together with the part *AB* of the element of the cone, and the length *Bf* of the tangent, determine it.

181. Any other curve, traced on the surface of the cone, will be obtained by the same method. In some instances it is even

possible to find a curve on the developement without constructing its projections. The intersection of the plane PQR (fig. 27) has been determined on the developement (fig. 29), by measuring, at once, the distances of its different points from the vertex V. The point L, for instance, comes in L', when the element VA, which contains it, is brought into a position parallel to the vertical plane; and consequently VL' is its distance from the vertex: VL, fig. 29, has been made equal to it.

EXAMPLE IV.

To find the intersection of two surfaces of revolution, the axis of which are in the same plane.

182. As yet the intersections of surfaces have been obtained by means of auxiliary planes: this last example will show the use sometimes made of auxiliary surfaces.

Let us take the vertical plane parallel to that of the two axes, one of which is supposed vertical.

Conceive, now, a sphere having its centre at the point of intersection A of the axes: it is evident, that this sphere will cut each surface of revolution in a circle; for, if, through one of the points common to the sphere and the surface, a plane be drawn perpendicular to its axis, this plane will cut the surface of revolution, as well as the sphere, in a circle which must be the same in both, since its centre is in their common axis.

Or, otherwise, if the generatrix of the surface and the great circle of the sphere are imagined to revolve together round their common axis; their point of intersection will describe a circle, which is, therefore, the intersection of the two surfaces.

183. Since then, a sphere, having its centre at the point A, intersects the two surfaces of revolution in two circles: these latter will cut each other on the surface of that sphere in two points common to both surfaces of revolution. Any number of such points may be obtained, by means of other spheres, and thereby the curve will be determined.

Let $BDCE$ be the vertical great circle of one sphere; this circle intersects the two meridian sections of the given surfaces in the two chords BC, DE, which are respectively the vertical projections of the circles of intersection of the sphere with the surfaces of revo-

Plate 11.

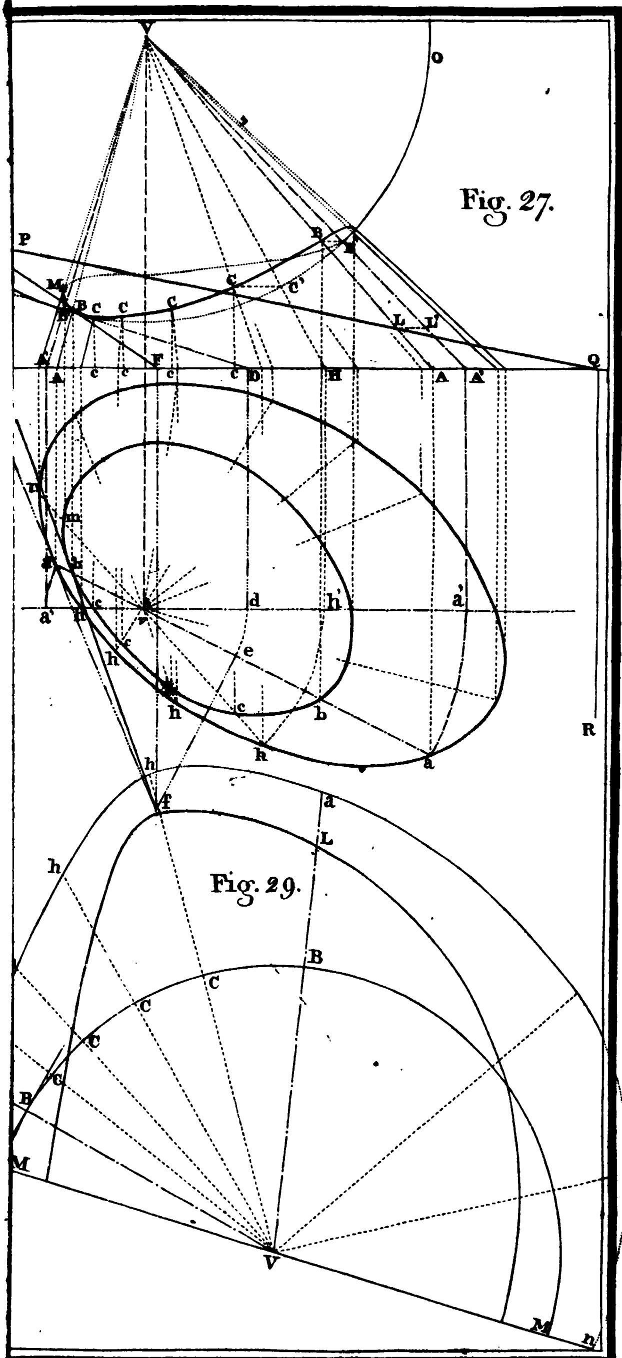

Fig. 27.

Fig. 29.

lution, since the planes of these circles are each perpendicular to its axis (59).

The horizontal projection of the circle, represented in the vertical plane by the line *BC*, is the circle *bmm;* the same projection of the other circle is not necessary to the operation.

The intersection *M* of the two chords *BC*, *DE* is the vertical projection of the two points of intersection of the circles projected in them: consequently, the points *m*, *m*, of the horizontal circle *bmm*, determined in it by the projecting line *M m m*, are the horizontal projections corresponding to the point *M*. They belong, therefore, to the horizontal projection of the curve.

184. To draw a tangent to the curve of intersection, is an operation, which has been performed so often by means of two tangent planes, that it would be unnecessary to notice it here, if this particular case did not furnish a very simple illustration of another method of drawing tangents to the curves of intersection of surfaces, both as general and as convenient, as that which we have heretofore employed.

This new method consists in using *normal* instead of tangent *planes*.

185. The *normal* to a surface is a perpendicular to the tangent plane: this line is, of course, perpendicular to the surface. For, the tangent plane is, in every direction, the prolongation of the infinitely small facet of contact. It follows from this, that any plane passing through a *normal* is perpendicular to the surface: it is therefore called *normal plane*.

186. If, now, two surfaces intersect each other in a certain curve, and if it be required to draw a tangent to it; construct through the given point of contact a normal to the first surface; and also another normal to the second surface: it is evident, that the plane, passing through them must be perpendicular to the tangent required; since this tangent, being separately perpendicular to each normal, must be at right angles with their plane.

The tangent will, therefore, be very easily obtained after having determined the normals.

In order to draw the required normals, at the given point of contact *M*, *m*, let us remark, that, since in a surface of revolution the tangent plane is perpendicular to the meridian plane, the normal must be contained in this latter: consequently it will intersect the axis. And, moreover, because of the perfect regularity of

such surfaces, it is evident that all the normals, drawn to a surface of revolution at the points of a same circle, pass through the same point of the axis.

187. Draw, then, the normal BF to the meridian curve of the first surface; it will cut the axis in a point F, through which the normal at the given point M must also pass (186), since the B points F of the generatrix and M of the intersection are in the same circle BC, bmm; the line FM is, consequently, the vertical projection of this normal.

In like manner, the normal EG to the generatrix of the second surface will determine in its axis the point G of the other normal; the projection of which will then be GM.

It is not, however, necessary to find the vertical projections of the normals; the plane which contains them may be constructed without it.

For, this plane must pass through the points F and G; consequently, the line FG is its trace on the vertical plane of the two axes; and, since this latter is parallel to the vertical plane, the trace FG is parallel to the vertical trace of the normal plane; the perpendicular MH, let fall from the point M upon this line FG, is, therefore, the vertical projection of the tangent. (186, 38)

Its horizontal projection may be obtained in various ways; and, among others, by drawing from the horizontal projection of the point of contact a perpendicular to the trace of the normal plane (38). It is clear, that the construction of this perpendicular requires only the direction, but not the real position of the trace. It will then be sufficient to obtain it in any horizontal plane whatever.

Let us use the horizontal plane BC, which passes through the point itself; and revolve this plane round its trace BC, on the meridian plane. The circle $BM'C$ and its ordinate MM' will determine the point of contact M' in its own horizontal plane. This point will, of course, be a point of the trace of the normal plane; but the point P, where its vertical trace FG intersects the diameter BC, is in the horizontal trace too (22): this latter is, therefore, the line $M'P$, to which drawing from the point m a perpendicular, we shall have the horizontal projection of the tangent.

The last part of the operation would be made more simple by projecting the whole of it upon the horizontal plane, instead of describing the circle $BM'C$: for the point P, common to the two accessory traces, being projected in p, the line mp, drawn from

this latter to the projection m of the point of contact, would be at once the trace, and, consequently, the line nm perpendicular to it, the tangent, required.

188. This method of drawing tangents succeeds even in some cases when the others would fail.

Fig. 31. Suppose, for instance, a horizontal cylinder intersecting a sphere ; both of them having their centre in the horizontal plane, it is evident, that at the point A of the horizontal plane, where the two surfaces intersect, the tangent planes to the cylinder and the sphere will be both vertical ; and consequently cannot, by their intersection, determine the horizontal projection of the tangent to the curve of the penetration of the two surfaces*: whereas the following observations on the normals will lead to it. Every normal to the sphere pierces the horizontal plane in its centre C; and every normal to the cylinder, in the point of its axis perpendicularly opposite to the point of contact. If, therefore, we draw through the point A the perpendicular AB to the axis of the cylinder ; the points B and C will be the points where the normals to the two surfaces, at their common point A, pierce the horizontal plane ; and, consequently, BC, the trace of the normal plane which passes through them.

The perpendicular AD to this trace BC is then the tangent required.

It may appear, at first, difficult to conceive the horizontal trace of a plane, which does not, in fact, differ from the plane of projections ; but the doubt will be easily dispelled, by imagining the operation to be executed upon a point at an evanescent distance from the real point A; a supposition which will convince the mind, without altering the result.

This case occurs frequently in practice.

We will here conclude the considerations on the intersections of surfaces, which belong to the elementary part of this treatise; and resume the determination of tangent planes,

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* This horizontal projection is a *parabola*.

CHAPTER VII.

OF PLANES DRAWN, THROUGH A GIVEN LINE, TANGENT TO A SURFACE.

189. The line through which the plane must pass being given, there remains only to determine the point of contact. This may sometimes be done by a direct method; but, more generally, it is necessary to use an auxiliary tangent surface, known to pass through the point of contact. The nature of the auxiliary surface or surfaces used depends on the data of the operation; in general, however, cylinders and cones are employed, on account of the facility they present in constructions.

190. Let us conceive a cylinder circumscribed to the given surface; it will touch it in a certain curve; and a plane tangent to this cylinder must also be tangent to the surface, since it coincides with the cylinder along the element of contact. But, this element having in general one single point in common with the surface, that must be the point of contact sought; it is evidently in the curve of contact of the given surface and that circumscribed to it; and consequently where the element of contact of the plane and cylinder intersects it.

191. Otherwise; draw, from a certain point of the given line, a cone tangent to the surface; the plane tangent to the cone will touch the surface, and, consequently, will be the solution of the problem, when the point of contact is not required; but, when it is the case, this point is found where the element of contact of the plane and cone intersects the curve of contact of this latter and the surface.

192. There is another method, more general yet, to obtain the point of contact: it consists in considering two points of the given line as the vertices of two cones, circumscribed to the given surface; since the curves of contact of the cones with the surface, each contain the point of contact, it follows that their intersection on the surface must determine it.

193. There will always be a number of contacts, and, consequently, of tangent planes passing through the line equal to the number of points of intersection of the two curves (or, in other words, to the number of tangent planes that can be drawn to each of the circumscribing ones.)

194. This solution requires the determination of the line of contact of a cone and a surface. The problem may be solved, in general, by drawing through the point assumed as the vertex of the cone a series of planes, each cutting the surface in a curve; the tangent to which, drawn from the vertex, is an element of the cone. This construction is, however, in a great many cases, made more simple, by some particular consideration, resulting from the nature of the surfaces which are the object of it. In the second part, and particularly in the applications to perspective and shades, we shall present many examples of the methods which are advantageous for each particular kind of surfaces.

We will present here only one example of tangent planes drawn through a line; because it is an elementary, and almost fundamental question.

To draw, through a given line, a plane tangent to the surface of a sphere.

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195. *1st. method.* By means of two tangent cones.

Fig. 32. Let *C*, *c*, be the projections of the centre of the sphere; *cb*, *CB*, its radius; and *ah*, *AH*, the given line.

Conceive through the centre *C*, the horizontal plane *CD*; it intersects the given line *AH*, *ah*, in the point *D*, *d*. If this point be taken as the vertex of a cone circumscribed about the sphere, the curve of contact of this cone and the sphere will be a circle(*) in a plane perpendicular to the axis of the cone, and consequently vertical; since this axis is a horizontal line passing through the vertex *D*, *d*, of the cone and the centre of the sphere.

But the two tangents *dl*, *dk*, to the horizontal projection of the great circle of the sphere, are evidently the two extreme elements of the tangent cone, and consequently the chord *lk*, the projection of its base or circle of contact.

(*) That the curve of contact of a cone and a sphere is a circle, perpendicular to the line, which joins the vertex of the former to the centre of the latter, may be thus demonstrated: draw from a given point a tangent to a circle, and suppose that the whole figure revolves round the line which passes through the point and the centre of the circle; then this latter will describe a sphere, and the tangent a cone; while the point of contact describes the circle, which is the curve of contact of the sphere and the cone.

196. Two planes might now be drawn tangent to this cone (81), and would give two solutions of the problem: since they would also be tangent to the sphere (191); but it will be better to determine their points of contact by means of another cone.

Let us therefore do with a vertical plane the same thing that we have done with a horizontal one.

Draw, through the centre *C*, *c*, of the sphere, a vertical plane parallel to that of projections; it will cut the sphere in a great circle projected in *EBF*, and the given line in a point, *g*, *G*: from this point describe a cone tangent to the sphere; its curve of contact will be a circle perpendicular to the vertical plane, because the axis of the cone is parallel to it. Consequently, for reasons similar to those given above (195), this circle will be projected on the vertical plane in the chord *EF*, which connects the two points of contact of the tangents *GF*, *GE*, to the vertical great circle *EFB*.

But the circles of contact of the two tangent cones, must, each, contain the two points of contact, which, consequently, are their intersections on the surface of the sphere (192).

And, since these points are in each of the two circles of contact, their horizontal projections must be somewhere in the projection *lk* of the first circle; and their vertical, in that of the second circle *EF*.

Consequently the line, which joins the two points of contact, and which is, of course, the intersection of the planes of the circles, is projected horizontally in *lk*, and vertically in *EF*. Let us find where it pierces the sphere; and we shall have the points of contact.

197. This line *lk*, *EF*, meets the horizontal plane *DB*, that passes through the centre, in the point, *N*, *n*.

Let us now conceive, that the plane of the vertical circle of contact turns round its horizontal diameter *lk*, to become itself horizontal: the two points of contact and the line which connects them, being carried round by it: this circle, in its new position, will be constructed by describing it on the diameter *lk*. The two points of contact remain to be determined in its circumference.

Let us then construct the line which connects them; the point *N*, *n*, of it remains fixed during the rotation of the plane, since it is in the hinge *lk*; we have, therefore, to determine only one point of the new position of the line. The point *O*, *o*, for instance, where it pierces the vertical plane *gb* of the centre, will, in the

revolution of the plane, describe round the hinge lk a vertical quadrant, the radius of which is its vertical altitude OT above the axis of rotation : it will, consequently, fall at a perpendicular distance ot, equal to OT, from the point o of the hinge lk. The point t is, therefore, the new position in the horizontal plane of the point T, o, of the line. If, then, a line be drawn through the points n and t, its two points of intersection P and Q with the circumference of the circle of contact, will be the two points of contact.

In order to find the projections of the same points, when in their real position, conceive that the plane returns to it, by revolving round the same hinge lk : the two points P and Q will, then, move in two vertical circles, the horizontal projections of which are the perpendiculars Pr, Qs to lk; consequently the points of contact are, in the horizontal plane, r and s.

Their vertical projections will be determined by projecting them on the line EF, in R and S.

198. The ordinates Rp, qS, must be equal respectively to rP and Qs : which may be also a means of construction.

199. The points of contact being known, the traces of the tangent planes will be found by drawing two parallels to the given line, each through one of the points ; and finding the points where they pierce the horizontal plane. The lines vnh and mhU, connecting these points n and m with the point h, where the given line pierces the horizontal plane, are the horizontal traces : the two others will be constructed by some of the known methods (199, 78).

200. The traces may also be obtained by drawing, from the points h and I, where the line pierces the planes of projections, perpendiculars to the respective radii, cr, CR, and cs, CS, of the points of contact; these perpendiculars must be the traces required ; since the tangent plane to a sphere is perpendicular to its radius (37).

201. It has most probably been remarked, that the principal part of the constructions has been executed in two planes parallel to those of projections, and passing through the centre of the sphere. This method of operating, in some more convenient planes than those of projections, is very often employed to simplify constructions. Many instances will be seen in the sequel, where some details of the operation, instead of being as here removed by projections into the general horizontal and vertical planes, are

performed immediately in two assumed planes of projections; making, as it were, quite a separate problem, the result of which being obtained, is then restored to the general operation; to this the secondary construction must always be connected by the natural disposition of the data.

202. If, for instance, as it is frequently the case, the general solution of a problem requires the determination of a tangent plane to a sphere; it is almost always advantageous to execute the operation in two secondary planes of projections made to pass through the centre of the sphere.

This problem: to draw a plane tangent to a sphere, recurring very often in graphical operations, requires a particular attention: it is therefore advantageous to examine it with all the modifications it may receive, and under the various shapes in which it may present itself. We will, accordingly, show here its solution, when the planes of projections pass through the centre of the sphere: a supposition which will greatly simplify the construction.

Solution of the same problem, and by the same method, when the planes of projections are assumed through the centre.

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203. Let *AB*, *ab*, be the two projections of the line; the point O of the base line being the centre of the sphere.

1*st Solution*. The points *A* and *b*, where the line pierces the planes of projections, are taken for the vertices of two cones, which touch the sphere in two circles; one vertical and projected in the line *cd*; the other perpendicular to the vertical plane and projected in *CD*: their two planes intersect each other in a line which passes through the points of contact. The intersections of this line, with the circle whose diameter is *cd*, are the points *E'*, *F'*, which are determined, by considering as fixed the point *n*, *N*, where the line pierces the horizontal plane, and consequently the axis of rotation *cd*; and finding the position *M'* where the point *M*, *m*, of this same line, in the vertical plane, falls in turning over the plane of the circle; the line, taking then the position *NM'*, intersects this circle in the points of contact *E'*, *F'*, which are brought back to their real position by the perpendiculars *E'e*, *F'f*, to the hinge; and then projected vertically in *E* and *F*. The whole of this is exactly the same as the preceding solution. The tangent planes would follow from the points of contact as above.

204. *2d Solution.* Through the given line, draw two planes tangent to one of the tangent cones, by finding (35) the point G where the line intersects the plane of the circle of contact; and drawing two tangents to it GE', GF', through that point. The two points of contact will thus be obtained.

In this solution, the points h, h, where the tangents intersect the fixed hinge cd, round which the circle of contact was turned down, will not stir when the base of the cone returns to its former position; and, since they are in the horizontal plane, they will, together with the point b, where the line pierces it, determine the two horizontal traces bh, bh, which, for a verification, must be perpendicular to the corresponding radii of the sphere.

205. *3d Solution.* The projections of the line, which joins the points of contact, being CD and cd; conceive through this line and the centre O a plane; it will cut the sphere in a great circle, which is intersected by the line in the two points of contact.

Cause then this circle to revolve round the horizontal line ON, drawn through its centre and the point N, where the line pierces the plane of projections, until it is laid down on the horizontal plane; it will, in that position, coincide with the horizontal great circle $cF''Ddc$: the line, moving round with the section of the sphere, will have a fixed point N in the axis of rotation; another point m, M, of the same line will fall at a distance $m'M''$ from the fixed axis, which is the hypothenuse (30) of a triangle, the base and altitude of which are mm', its distance from the hinge ON, and mM, its vertical height.

The line NM'', now determined, cuts the great circle in the points E'', F'', which must revolve back, to their real position, in two vertical circles projected in the perpendiculars $E''e$, $F''f$, to the axis of rotation.

The intersections f and e of these perpendiculars, with the projection cd of the hinge, are the same horizontal projections of the points of contact before found.

We have dwelt much more on this question than on all the preceding, because the sphere is to surfaces what the circle is to plane figures. We will now show the solution of the same, by means of a circumscribed cylinder; not only as an illustration of the general mode of solution; but, also, that, in every particular case, the operations in which spheres are used may be performed by the method best adapted to the nature of the data.

206. *2d. Method.*—Let us conceive now a cylinder tangent to the sphere, and parallel to the given line: it is evident that their curve of contact is a great circle perpendicular to the cylinder. The construction of this curve of contact can, consequently, be obtained, at once, by drawing through the centre of the sphere a plane perpendicular to the cylinder, or, which is the same thing, to the given line.

But, since the points of contact are in the great circle of contact, if we conceive the two tangent planes to be constructed, the plane drawn through the centre, will intersect them in two tangents to the section made in the sphere; and these lines will meet at the point of intersection of the same plane with the given line.

Therefore, if, inversely, the two tangents to the great circle are constructed, the tangent planes will follow.

Fig. 34. Draw now through the centre O the perpendiculars OD, Oc, to the projections AB, ab, of the line; they will be the traces of the plane perpendicular to the given line.

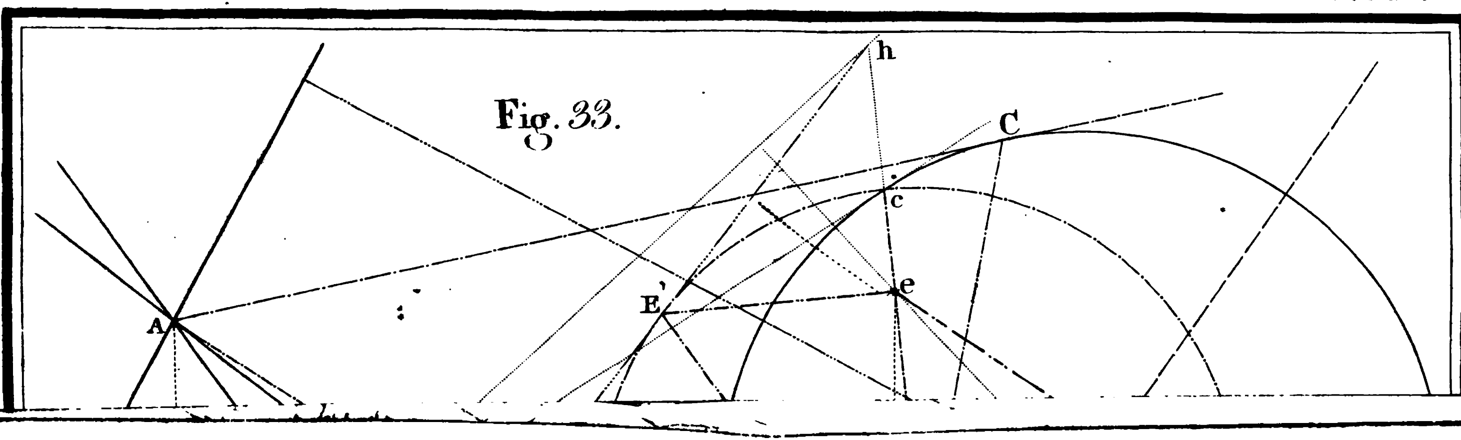
This plane cuts it in a point E, e, (35) and the sphere in a great circle, in order to draw two tangents to which from the point E, e, let us revolve it round the horizontal trace Oc of its plane. When this circle is in the horizontal plane it coincides with the great circle of the sphere; but the point E, e, falls at a perpendicular distance cE' from the hinge, equal to the hypothenuse of a triangle, whose base and altitude are ce and ET; drawing, then, from the point E', the two tangents $E'H'$, $E'F'$, the points of contact H', F', are determined; and there remains only to revolve them back to their real position, in order to obtain their projections.

207. But let us, first, remark, that if only the tangent planes were required, their horizontal traces might now be drawn through the point b, where the given line pierces the horizontal plane, and the points, g, g, where the tangents intersect the fixed axis cO.

208. Since then the point g of the tangent $E'H'$ remains fixed, when the plane returns to its natural position, it follows, that egh is the horizontal projection of this tangent: but its point of contact is evidently projected somewhere in the perpendicular $H'h$ to the axis of rotation (29), it is, therefore, at the intersecton h of this line, and the tangent egh.

209. If the other tangent $E'F'$ intersected the axis of rotation too far, the projection of its point of contact might be found by means of the line which connects the two points of contact. This

Plate 13.



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line $H'F'$, in its rotatory motion, constantly passes through the point n, where it cuts the fixed axis: its horizontal projection, which must then pass through this point n is consequently hnf; and its intersection f with the perpendicular $F'f$, the projection of the other point of contact.

In order to find the vertical projections of the same points, let us remark, that the vertical projection of the point g is G; and, consequently, that of the tangent egh is EGH, the intersection of which, with the projecting line hH, gives the vertical projection H of the first point of contact.

210. In like manner, would be found the vertical projection FNH of the line which joins the points of contact; and therefrom the vertical projection F of the second point.

211. It may farther be observed, that the line which joins the points of contact, being perpendicular to the plane drawn through the centre of the sphere, and the given line AB, ab, the projections FH, fh, must be perpendicular to the corresponding traces Ob, OA, of that plane. This remark will serve as a means of construction, or of verification.

212. The tangent planes will be deduced from the tangents or normals, by the methods already known.

213. The principal remarks, which may lead to the determination of the traces are: 1st That they must pass through the points where the given line intersects the planes of projections. 2d That they are perpendicular to the radii of the points of contact. 3d. That the horizontal ones contain the fixed points of the tangents in the axis of rotation. 4th. That the two traces meet in the common intersection. 5th. That the planes contain the lines drawn through the points of contact parallel to the given line.

214. This last problem completes a series of propositions that are all necessary to the solution of the principal questions relative to high Geometry. They compose the elementary part of Descriptive Geometry, which we will conclude by a few words on *oblique projections.*

It is sometimes advantageous to project certain parts of an operation by oblique, instead of perpendicular lines, and thereby to avoid the introduction of new planes of projections. This will be explained by the following example.

Fig. 35. Suppose that three lines AB, ab; CD, cd, EF, ef, being given, it be required to find many elements of the surface

generated by a fourth line, moving so as to touch them in all its positions.

It is evident that, if, through a point of the first line *EF*, *ef*, for example, and through the second line *CD*, *cd*, a plane be made to pass, it will intersect the third line in a point, which being joined with the assumed one in the first, will give an element of the surface; since this last line passes through two points, situated in the first and third directrices, and is contained in the same plane with the third, which it must consequently intersect. The determination of such an element would, by the ordinary method, require the construction of a plane passing through a point and a line; and also its intersection with another line: but the operation may be simplified, by previously projecting the three directrices, upon the vertical plane, by oblique lines parallel to one of them *EF*, *ef*, for instance: this latter line will, of course, be projected in the point *E* where it pierces the vertical plane.

The same points, *C* and *A*, of the two other lines will be their own projections; and their points *D*, *d*, and *B*, *b*, being projected upon the vertical plane by the lines *DH*, *dh*, for one, and *BG* and *bg*, for the other, will be found at the points *H* and *G*, where these oblique lines of projections meet the vertical plane.

The oblique projections *CH*, *AG*, of the two last directrices being constructed, the determination of an element of the surface will be a very easy operation: for, any line, like *EI*, passing through the point *E*, is the oblique projection of an element; since all the points of *EF* are projected in that point *E*.

The orthogonal projection of the same line will be deduced from the line *EI* by means of the oblique lines of projections *Ii*, *Ll*; which cut the vertical projections of the directrices in the points *i*, *l*, through which the element *ilm* is then drawn. Its horizontal projection *jnm* will be found in the usual manner, and, if the operation is correct, the points *j*, *n*, *m*, must be in the same right line.

The point *O*, where the orthogonal and oblique projections intersect, is that where the element pierces the vertical plane; it must, consequently, correspond to the intersection *o* of the ground line and the horizontal prejection *jnm* of the element.

All the points like *o*, where the elements meet the vertical plane, would give the curve of intersection of the surface and the vertical plane.

Plate 14.

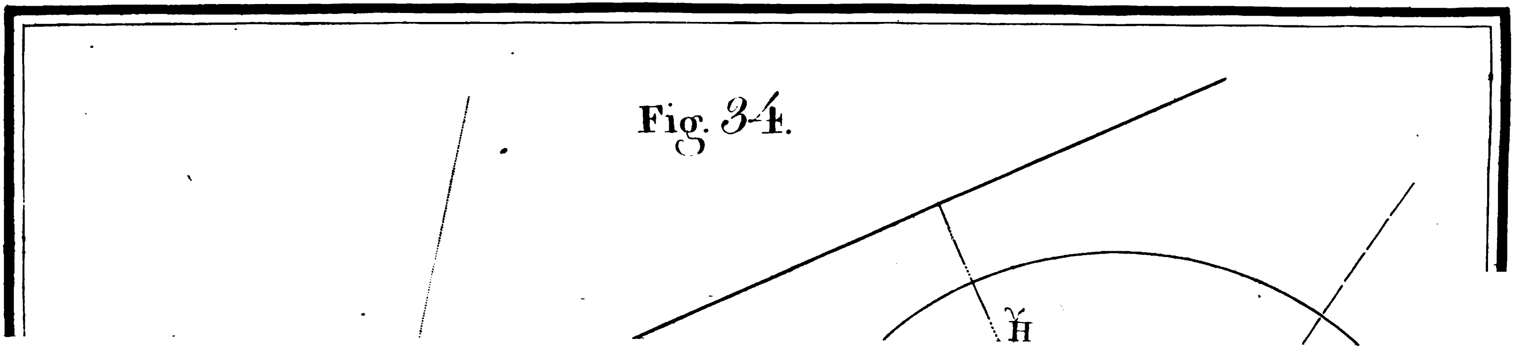

Fig. 34.

The advantages of oblique projections, in the determination of a great number of the elements of the above surface, are sufficiently striking to recommend their use in many circumstances.

215. This elementary part of descriptive geometry is sufficient for the solution of almost every problem; but, as, in every case, great simplifications may be introduced into the operations, by a judicious choice of the planes of projections, or a skilful arrangement of the different parts of the solution, which require a great habit of geometrical conceptions; many simple and useful applications have been added to this first part, as an elementary introduction to the higher and more complex considerations of the second.

CHAPTER VIII.

Solution of some Geometrical Problems.

PROBLEM I.

To find the centre of a sphere that must pass through four given points.

Or otherwise: to circumscribe a sphere about a triangular pyramid.

216. In order to solve this problem, let us remark, that, each side of the pyramid is a chord of the sphere; and that, consequently, a plane drawn through its middle point, perpendicular to it, must pass through the centre of the sphere.

This centre, therefore, will be determined by drawing, through the middle of three sides of the pyramid, and perpendicular to them, three planes, which will intersect each other in a common point, equally distant from the extremities of the sides, and, consequently, from the angles of the pyramid.

It is, indeed, obvious, that the first plane is every where equally distant from the first side; and that the second plane is similarly situated with respect to the second side: therefore, any point of their line of intersection is at the same distance from the extre-

mities of the first and second line: the same thing being applied to the third plane, would show that its intersection with the line common to the two first, is the centre of the circumscribed sphere; since it is equally distant from the extremities of three sides of the pyramid, or, in other words, from the four given points.

217. Let us now proceed to the graphical construction of this solution.

As the position of the planes of projections is arbitrary, let us assume for the horizontal one the plane determined by any three of the points; they form a triangle which may be taken for the base of the pyramid: then, the fourth point being the vertex, let us take the vertical plane parallel to one of its sides.

Fig. 36. Let *abc* be, in the horizontal plane, the triangle formed by three of the given points; and *d*, *D*, the projections of the fourth, the line *AD* being parallel to the vertical plane.

It is now evident that the planes perpendicular to the lines *ab*, *bc*, *ac*, are vertical: if, then, the three perpendiculars *eo*, *of*, *og*, are drawn through the middle of each of these, they will be the horizontal projections of three vertical planes passing through the centre. It is, however, to be remarked, that the three perpendiculars intersect each other in the same point *o*, which is well known to be the centre of the circle circumscribed about the triangle *abc*; this circle is here a small section of the sphere. One of those perpendiculars is, therefore, unnecessary, and must only be considered as a means of verification: and, since the projections *of*, *og*, *oe*, of the vertical planes, intersect at the same point, it follows that, the planes themselves cut each other in a vertical line, which must pass through the centre of the sphere, the horizontal projection of which is then the point *o*.

From all this we must conclude, that the three perpendicular planes ought to be drawn to three sides, that are not situated in the same plane.

Let us, consequently, draw the last plane perpendicular to the side *ad*, *AD*. Because this line is parallel to the vertical plane, the plane perpendicular to its middle *H* shall be projected in the perpendicular line *Hh*, in which, therefore, the vertical projection of the centre must be found.

But this same point is also contained in the vertical intersection *Oo*, previously determined by any two of the three above mentioned perpendicular planes; consequently, the meeting point *O*

of the lines Oo, Hh, is the vertical projection of the centre of the sphere.

The radius of the same is measured by any one of the lines drawn from the centre O, o, to the given points; the line OC, oc, for instance, the length OR of which being constructed (19), the vertical and horizontal projections of the sphere can be described.

PROBLEM II.

To determine the centre and radius of a sphere, that may be tangent to four given planes.

Or else: To inscribe a sphere in a given pyramid.

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218. Because the sphere must touch the four faces of the pyramid, a plane drawn through its centre and any one of the six sides, must bisect the angle of the two faces which intersect in that side. If, then, three sides, which do not pass through the same vertex, be chosen, and the angle formed by their adjacent faces bisected by three planes; these latter will have a common point of intersection, equally distant from the faces of the pyramid, and which shall, therefore, be the centre of the inscribed pyramid.

Although a pyramid has six sides and six faces, only three planes are necessary to the solution of this problem: the three others might be proved to be here, as in the last question, mere verifications of the operation.

To simplify the construction, we shall here take the triangular base in the horizontal plane.

Fig. 37. a, b, c, d, being, then, the horizontal, and A, B, C, D, the vertical projections of the angular points of the pyramid; draw, from the vertex D, three vertical planes perpendicular to the sides of the base: they will be projected in the three lines dh, dg, df. Each of these planes intersects the base and the adjacent face of the pyramid in two lines that contain their angle. In order to determine the three angles thus formed, lay off from the point I of the vertical line Dd, the distances IH, IG, IF, respectively equal to dh, dg, df, and draw, from the vertex D, the lines DH, DG, DF: they will form, with the common intersection AC, angles equal to those of the corresponding faces of the pyramid with its base: these angles can now easily be bisected.

But the three planes, which divide equally the angles of the

faces and the base, form on this base a pyramid of which the vertex is their common point of intersection, (or centre of the sphere,) if then this second pyramid is constructed, the centre shall be determined.

In order to find its vertex, draw, at any height whatever, a horizontal plane *MN*; it will cut this second pyramid in a triangle similar to the base, each side of this triangle will have a point in the corresponding bisecting line. This point, in each of them, will be determined by the intersection of the horizontal plane *MN* with it.

These three lines Ff', Gg', Hh', being cut by the line *MN* in the points f', g', h'; draw from them the perpendiculars $f'P$, $g'Q$, $h'R$, to the base *AC*; and carry the three distances *FP*, *QG*, *RH*, on the corresponding perpendiculars *fd*, *gd*, *hd*, and then the points *p*, *q*, *r*, will each belong to one of the sides of the intersection, made in the second pyramid by the horizontal plane *MN*.

Since this triangle is parallel to the base, draw the parallel lines *sv*, *tv*, *ts*, through the points *p*, *q*, *r*, and they will form the new triangle *stv*, the angles of which *s*, *t*, and *v*, being evidently in the sides of the new pyramid, will determine them by being joined with the corresponding angles of the base *a*, *b*, *c*.

The common point of intersection *o*, of the horizontal projections of these three sides, is the projection of the vertex of the pyramid, or centre of the sphere.

Its vertical projection *O* is, at the same time, in the projecting line *Oo*, and in each of the vertical projections *AS*, *BT*, *VC*, of the sides of the new pyramid: one of these lines will be sufficient to determine it.

As to the radius of the sphere, it must evidently be equal to the vertical altitude *OK* of its centre.

The different points of contact, one of which is *K*, *o*, would be easily obtained, by demitting from the centre perpendiculars upon the tangent planes.

PROBLEM III.

To determine the vertex of a pyramid, the base of which is given, as well as its other three sides.

Fig. 38. 219. This problem is the same as: *to find the position of a point, the distances of which from three fixed points are given.*

Let *T*, *U*, *V*, be the given distances; take for horizontal plane the base *abc* of the pyramid; and assume the vertical plane in a direction perpendicular to one of the sides of the base; to *ab*, for example.

Conceive a sphere to be described from the point *a*, with a radius equal to *T*, it will contain the vertex of the pyramid; a second sphere, having its centre in *b*, and the line *V* for its radius, would contain it too: their vertical circle of intersection, projected in *de*, must therefore pass through that vertex; which will then be perfectly determined by the intersection of this circle *de* with a third sphere, described from the centre *c* with a radius *U*.

But the circle *hg*, common to the first and third sphere, must also pass through the vertex; and, since it intersects, on the surface of the sphere *a*, the first circle *de* in two points projected in *m*, this point *m* is the horizontal projection of the two answers to the question: the one above and the other below the horizontal plane of the base.

Their vertical projections are the intersections *M*, *M*, of the line *mM*, and of the projection *DME* of the circle, projected horizontally in *de*.

The nature of the question must teach which of these is the true answer.

Many other problems relative to the pyramid might be proposed: we have here presented the most simple among them.

PROBLEM IV.

To determine the position of a point, the distances of which to four given points are proportional to given quantities.

220. The curve which has the property that the lines, drawn from any of its points to two fixed points, are in a constant ratio, is

well known to be a circle, the diameter of which passes through the given points.

The construction of this circle is as follows: Divide the line *AB* (fig. b). in two parts, bearing to each other the given ratio; the point of division *C* is evidently in the curve. From *C*, with a radius equal to the smaller division *BC* describe a circle; and, from the other point *A* of the line, draw a tangent to it, which will intersect the perpendicular *BD*, to the extremity *B* of the given line, in a point *D*, the perpendicular to which $DO = OC$ is the radius of the circle.

221. Conceive, now, that the circle revolves round its diameter, which is the given line *ABO*; it will generate a sphere, all the points of which will be at the same proportional distances from the fixed points *A* and *B*.

This property of the circle, then, extends itself to the sphere.

222. Now, to return to the solution of the problem, let us suppose that the distances of the required point to the first and second given points, are to each other in the ratio of *m* to *n*: in consequence of what has just been remarked, it will be situated in the sphere, which is the locus of all the points whose distances are in this same ratio of *m* to *n*.

In like manner, the ratio of its distances to the second and third points would furnish another sphere, in which it must also be contained.

Lastly, a third sphere would be determined by its relation to the third and fourth points; and then the point of intersection of these three spheres would be the answer to the question; which is, consequently, a modification of the preceding.

The only difference is the previous determination of the centres of the spheres, which must be effectuated by forming two triangles with the four points, and revolving one of them over, upon the plane of the other: after this operation it will be easy to find, by the above construction (214), in the lines which join the given points, the centres and radii of the circles which correspond to the spheres.

The two triangles, being then replaced in their relative position, the respective situation of the centres will follow; and, with a proper choice of the planes of projections (213), the problem will be reduced to the preceding one.

We leave the graphical construction as an exercise for the reader.

This question is the same as the following, which was proposed a few years since in a Scientific Journal: *to determine the position of a globe observed from four points, by means of its apparent diameters.*

For, the distances from the points to the centre of the globe, are proportional to the secants of four known angles; and, consequently, in known ratios.

Spherical Trigonometry.

223. The questions which may be proposed on spherical triangles, can all be solved by means of the triangular pyramid: for, (fig. *m.*) the centre of the sphere, on which a triangle is traced, may be considered as the vertex of a pyramid, the sides of which are the radii passing through the vertices of the triangles; the angles included between these radii, and which are measured by the sides of the triangle, are the faces of the pyramid.

The problems on the pyramid, which, in general, have for object the determination of its angles and faces, are particularly useful in architecture and carpentry: they must, for this reason, naturally precede the applications to these important branches.

224. A spherical triangle contains three angles and three sides: out of these six parts, three are sufficient to determine the triangle. There may be proposed, therefore, six problems on them which we will solve successively.

SOLUTION OF THE FUNDAMENTAL PROBLEMS OF SPHERICAL TRIGONOMETRY.

PROBLEM I.

The three sides of a spherical triangle being given, to find the three angles.

225. The pyramid, which corresponds to the given triangle, (223) being spread out on a plane, let $c'Sa$, aSb, bSc, be its three faces.

Now, suppose the two extreme faces to turn round the fixed

sides Sb, Sa: the extreme sides Sc, Sc', of the moving faces will describe two cones; the line of intersection of which, will be the third side of the pyramid in its real position.

Let us consider in the lines Sc, Sc' two points F and A, equally distant from the common vertex S of the two cones; it is evident, that these points must come in contact, when the two generatrices Sc, Sc' meet each other.

But the point F describes, round the fixed side Sb and in a plane perpendicular to it, a circle which projects itself, upon the fixed face aSb, in the line FE perpendicular to the axis of rotation Sb: in like manner, the point A of the line Sc' describes, round Sa, another circle projected in ABI. The circumferences of these circles must cut each other, since they are at equal distances from the vertex S: consequently, the point D is the projection of their intersection; and SDd that of the side of the pyramid.

226. If, now, the two circles are turned down upon the horizontal plane round their diameters ABI, FED; or, in other words, if the circles AGC, FHC' are described from the centres B and E, the vertical intersection of their planes will coincide, in each of them respectively, with the ordinates DC and DC': consequently, C, and C' shall be, in these circumferences, the common point in which they intersect. And, as this point is in the third side of the pyramid, CBD, $C'ED$, are the angles adjacent to the fixed face.

227. The third angle might be determined by supposing one of its adjoining faces fixed; and performing with the two others the same operation as above; but it will be more simple to construct it by means of a plane perpendicular to the third side projected in Sd; the two lines, along which it intersects the two collateral faces, measuring that angle. But, when the pyramid is spread out, the situation of the point projected in D, through which the perpendicular plane is drawn, being, in one face, the point A, in the other, the point F; it follows, that AG, and FH, respectively perpendicular to Sc' and Sc, are the intersections of the plane with the collateral faces; its trace, on the fixed face, is, therefore, the line GH. The construction of the triangle, formed in the pyramid by the lines GH, HF, AG, will give, as is represented in the figure, the third angle sought GKH.

228. The verifications of the operations are $CD=C'D$; $CI=IF$: and SDd perpendicular to the trace GH (35): besides the vertex K must be found in the line SDd (27).

229. It may be remarked: that, in the circumference described with the radius $SA=SF$, the arcs, contained within each of the faces, are the sides of the spherical triangle corresponding to the pyramid.

230. The construction shows, that the question would be absurd if the angle of one of the faces were greater than the sum of the two others.*

PROBLEM II.

Having given in a pyramid two faces and the contained angle, to find the other parts.

Fig. 2. 231. Let $c'Sa$, aSb, be the two given faces spread out on the same plane: from a point A of Sc' draw a perpendicular AB to Sa; and draw through the point B the line BC, making with this perpendicular an angle CBI equal to the given one.

The angle CBI being supposed to stand in a vertical plane, conceive the point A to be turned up round the side Sa and thus to come to the point C of the side CB of the given angle. This last point determines the real position of the side Sc'. If now, through the vertical CD, drawn from the point C, a plane be directed perpendicularly to the next side Sb, it will contain the triangle $C'DE$ formed with the base DE and altitude $C'D=CD$ (228) and in which the angle $C'ED$ is that of the fixed, with the third face of the pyramid.

Produce then the base DE of this triangle; and lay off on it from E to F the line EF equal to its hypothenuse $C'E$; and the line SF will determine the third face bSc.

232. Since the point F must, in closing up the pyramid, coincide with the point A; it follows that SF must be equal to SA.

Besides, because they must meet at the point C, the line $CI=IF$.

These remarks may serve either as verifications, or means of construction.

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* It is easy to perceive, that this problem is the graphical *reduction of angles to the horizon*: for, an observed angle, and the two angles of depression, are the three faces of a pyramid, one of the diedral angles of which is the reduced angle.

The other lines of the figure correspond to the supposition that the supplement of the given angle CBI makes part of the data, as well as the angle itself; and furnishes therefore a second answer to the problem: as is frequently the case in trigonometry.

233. As to the other parts of the pyramid or spherical triangle, they would be found as in the first problem; since the three faces are now known.

PROBLEM III.

Knowing in a pyramid two faces and one of the opposite angles, to find the other face.

Fig. 3. 234. Let aSb, bSc, be the given faces, and $H'eE$ the angle formed by the face aSb and the unknown one.

If the plane of the angle $H'eE$ be considered as vertical, the lines $H'e$ and eS will be the two traces of the plane of the third face; the position of which is thereby determined.

Suppose, now, that the face opposite the given angle revolves towards it, round the side Sb; its other side Sc will describe a right cone, the base of which Ff' is in a plane $FEDG$ perpendicular to the hinge Sb: the two lines of intersection of this cone with the plane of the third face will give two answers to the question.

In order to find the two lines just mentioned, we have to construct the intersection of the base of the cone with the plane $H'eS$. But the trace of the plane of this base, on the vertical plane of the given angle, being the vertical line EH', the point H' is both in the plane of the base of the cone, and in that of the face sought. This vertical EH', being turned down with the plane of the base, will fall on EH; and consequently H is a point of the trace of the plane of the face, on that of the base $f'fF$; and, as G is evidently another point of it, the trace itself is the line HG, which cuts the base of the cone in two points f, f'.

Consequently; when the side SF is in the plane of the unknown face, its point F may be either in f or f': let us suppose it in f, which project down in D.

In turning down, upon the known face aSb and round the side aS, the plane $H'eS$ the distance between the fixed point G and the moving one f will remain the same; consequently, the latter will fall somewhere on a circle fA described from the centre G, with a

radius fG: but the same point revolves round the fixed line Sa, in a perpendicular plane projected in the line DBA; the intersection A of the arc fA and this last line is, therefore, in the developement, the position of the point f: and ASa the third face required.

235. As a verification, the line SA must be equal to SF; since, in their real position, they are one and the same line.

To complete the pyramid, it only remains to construct, as above (231), one of the angles: for, one of them HeE is given; and the other fED results immediately from the operation.

236. The construction of the second solution f' of the problem does not differ from the first.

237. The three foregoing solutions would be sufficient for the six cases of the triangular pyramid: the three last might be reduced to those before explained, by means of the supplementary pyramid, formed by drawing from a point within the given pyramid a perpendicular to each of its faces: the angle included by two of these perpendiculars is evidently supplemental to that of the faces with which they are at right angles. (fig. n).

238. But the assumed point may be considered as the vertex of a pyramid, the three sides of which are the perpendiculars drawn to the faces of the first; two of these perpendiculars, containing a face of the new pyramid, it follows that the angles round its vertex are the supplements of the diedral angles of the other pyramid. Consequently, in solving a problem relative to it, the faces may be converted into angles, and vice versa, by means of the supplementary pyramid.

We will, however, give a direct solution of the three last cases.

PROBLEM IV.

Two angles being given and the face adjacent to them, to construct the pyramid.

Fig. 4. bSc'' is the face; and Cbd, $C'ed'$ the angles.

239. Conceive a plane parallel to the face; it will cut the unknown faces in two lines parallel to the lines bS, Sc''; and, the two sides of the given angles at the same height.

From a point C, in the line bC, draw the parallel CD to bS; and then, from a point C', situated in the line $C'e$ at a height $C'd'=Cd$,

draw likewise a parallel *C'D*. It will evidently intersect the first in a point *D* of the third side of the pyramid.

Let, now, the two planes *SBbC*, *SEeC'* be turned over, round their respective traces, *bS*, *Se''*; the point of the third side projected in *D* will fall in the perpendiculars *DBA*, on one side, and *DEF* on the other. Its position is determined in them, by making *BA* and *EF* respectively equal to *bC* and *eC'*; or, else, by the circle *AF*, described with a radius equal to the hypothenuse of a triangle, having *DS* and *Cd* or *C'd'* for its base and altitude.

PROBLEM V.

Having given two angles and one of the opposite faces, to construct the other faces.

Fig. 5. 240. *BSD* being the giv n face, and *CBD* its angle with the plane *CBS* of the second; the solution consists in drawing through the side *SD* a plane which may make with that of the second face a known angle *BC'D'*.

Taking, therefore, the point *D* of the line *SD* for the vertex of a right cone, described by a line making with the opposite plane an angle equal to *BC'D'*; it is evident that a plane drawn tangent to it through the side *SD* must also make the given angle with that of the other face.

But the perpendicular *DL* to the trace of the opposite face, is also perpendicular to the face itself, since it is situated in the plane *C'BD*; and will, therefore, be the axis of the auxiliary cone, the element of which, situated in the vertical plane *C'BD*, is the line *CD* parallel to *C'D'*; the radius of its base is then *LC*.

Let now the plane of the face which contains the base be revolved over round its trace *BS*; the centre *L* of the circle will fall in the point *L'* of the perpendicular *DBL'*. From this point with a radius *L'C'*$=$*LC* describe a circle, and draw to it the tangents *SA*, *SA'*; they will be the two traces, in the plane of the second face, of two planes which make with it the required angle.

Therefore, if there be no condition to particularize the question, *ASB* and *A'SB* will equally well answer for the second face situated in the plane *C'BS*.

Having got now two faces, the solution of the problem may be completed as in the preceding cases.*

PROBLEM VI.

Having given the three diedral angles of a pyramid to find the faces.

Fig. 6. 241. Let A, B, C, be the three given angles: a line Bb, being taken for the ground line, draw the line bc making with it the angle cbG equal to one of the given angles, A, for example; this line bc may be considered as the vertical trace of a plane making with the horizontal one an angle equal to A.

If now a third plane could be drawn, so as to make, with the assumed ones, angles equal to B and C, it would, together with them, form the pyramid required.

Let then any point D be assumed within the angle cbG; and two perpendiculars DE, Dl, demitted from it upon the lines bG and cb; this point D may be taken as the common vertex of two cones, of which DE and dl are the respective axes; the elements

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* If the given angle $D'C'B$ was a right angle, the lines DC and DL would coincide, and the side Sc' of the second face would pass through the point L' (fig. 5. bis). this remark will serve to prove the following important theorem, that: *if two planes at right angles with each other move between the two sides of a given angle, so as to pass each through one of them, the surface generated will be an oblique cone.*

For let BSD be the given angle, LB being the trace of one of the planes in one of its positions; LD will (from what precedes) be the trace of the other; and the line passing through the points L and S, which is projected in MS their intersection.

But two other positions Bl and Dl of the planes would give another point l of intersection; and as all points, like L, l, must be in the semi-circle $BLlD$, it follows, that this circle is the locus of all the points of intersection of the traces of the rectangular planes, and consequently the base of a cone having its vertex in S, since all the intersections of the planes must pass through this point S, and the points L, l, &c.

As the vertical plane might as well be taken in the direction BB' perpendicular to SD. It follows (since the reasoning would be the same) that the cone has two circular bases; and that their planes are respectively perpendicular to the sides SB, SD, of the fixed angle.

They are called the *subcontrary* sections of the cone.

of each making with the plane upon which it stands an angle equal to one of the two other given angles: these elements are, in the vertical plane, *DG* and *DC*.

It is very evident, that a plane tangent to these cones must make with the two planes on which they stand two angles equal to the given ones; the construction of this tangent plane will consequently solve the problem.

242. In order to draw a common tangent plane to the two right cones, we will remark, that a plane which touches a cone must also touch a sphere inscribed in it (191).

If, therefore, we suppose two spheres inscribed, each in one of the cones, the plane drawn through the vertex *D* must be tangent to them as well as to the cones. And the problem will be again modified and reduced to *drawing through a given point a plane tangent to two spheres.*

243. Let us first construct them: at the point *G* draw a perpendicular *GH* to the element *DG* of the cone; the point *H*, where it cuts the axis, will be the centre, and *GH* the radius of the sphere tangent to the first cone. The other might be found in the same way: but, as it is more simple to take it equal to the first, we will determine it so.

At the vertex *D*, and to the side *DC* of the second cone, draw the perpendicular *DK* equal to the radius *HG*; and, from its extremity *K*, the parallel *KC*, to the axis of the same cone; the intersection *C* of the side *DC* and this parallel, determines the position of the second sphere, of which the perpendicular $CM = DK$ is the radius and *M* the centre.

To facilitate the constructions, we will, instead of the plane *cbs*, take the parallel one *CBS*, drawn through the point *C*, for that of the face of the pyramid; and, consequently, consider *CL*, instead of *cl*, as the base of the second cone.

In order now to draw, from the point *D*, a plane tangent to the equal spheres, we will conceive a cylinder circumscribed about them; this cylinder must also be touched by the tangent plane; and, consequently, its circle of contact with each sphere must contain the point of contact of the plane with the same (190). But this point, in each sphere, is besides on the small circle in which it is touched by the cone circumscribed to it.

Therefore, the circles of contact of each sphere with the cylinder

and with the cone will intersect in two points, which shall be the contacts of the tangent planes common to all these surfaces.

But the circle of contact of the first sphere and the cylinder is perpendicular to its axis, and consequently projected on the vertical plane in a right line *HO;* the circle of contact with the cone is, likewise, projected in a straight line *EOG*. And, therefore, the point *O* is the common projection of their two intersections; or, what is the same, of the two points of contact in that sphere. In like manner, the point *P*, where the projections *CL*, *MP*, of the corresponding circles intersect on the other sphere, is the common projection of its two points of contact with the tangent planes.

It remains, now, to determine, from the points of contact, the intersections of these planes with the plane of the horizontal face and that *CBS* of the second. But, as the construction would be the same for both, we will only show it for one.

244. Those intersections or traces of a plane tangent to the spheres, or rather to the cones, must, of course, be tangent to the bases of these latter surfaces.

Describe, therefore, in the horizontal plane, with its radius *EG*, the base of the first cone; and, by means of the ordinate *OF*, determine in it the point of contact *F*, to which draw the tangent *FS*, which, being the first intersection required, determines the horizontal face *BSF* of the pyramid.

In order to construct the intersection with the oblique plane *CBS*, revolve it over round its trace *BS*. The center *L* of the base of the second cone will fall in *L'*, from which, with a radius *L'C* = *LC*, describe the circle of this base. Draw, then, a tangent *AS* to it, from the vertex *S*; and *ASB* will be the second face. BS

The other parts of the pyramid we know how to construct.

The verifications are here: 1st. That the point *P*, being revolved in *p*, the ordinate *pA* must pass through the point of contact *A*.

2d. That the line *PO*, which passes through the two points of contact *P* and *O*, and is, consequently, the element of contact with the cylinder, must be parallel to its axis *MH*.

245. The construction for the other tangent plane would be exactly the same, and would, besides, give a perfectly symetrical pyramid.

If instead of a cylinder tangent to the spheres, a cone, having its vertex between them, had been employed to determine the

tangent planes, the result would have been two other symetrical pyramids, having two of their angles equal to the corresponding two of the other solution; but differing in the last angle, which must have been supplemental to the third angle of the same.

The construction of this case so much resembles the first, that it has been thought unnecessary to make a figure for it.

Although the solution of this problem would be much easier by means of the faces of the supplemental pyramid, the foregoing is more interesting, as it presents an example of the manner of using auxiliary tangent surfaces.

We shall not extend farther, what, in this first section, relates to descriptive geometry. The second part will contain, besides the complement of the general principles of this science, its application to shades, shadows, and perspective; a very interesting subject, which will be made extremely simple by the methods of descriptive geometry; it is, indeed, easy to understand, that shades and shadows being determined by rays of light circumscribing the different bodies to be represented, require only the construction of the contacts and intersections of cones or cylinders.

As to perspective, it is the determination of the contours of objects, by means of lines passing through them and the eye of the observer; the intersection of these lines by a plane is the picture required; consequently, the questions of perspective depend all on one of the most elementary problems of Descriptive Geometry: the intersection of a cone by a plane.

THE END OF THE FIRST PART OF DESCRIPTIVE GEOMETRY.

1st Section

Finished Descriptive Geometry

March 10th 1826

Conic Sections.

DEFINITIONS.

1. If the lengths of the perpendiculars PC, $P'C'$, to the fixed line AX (fig. 1) be given, and also the distances AC, AC', from the foot of each perpendicular to a fixed point A of the line AX, then the relative position of the points P and P' will be perfectly determined.

Consequently, if all the points of a curve can be determined in that way, the curve itself will be completely defined.

2. The perpendiculars PC, $P'C'$, &c. are *ordinates* of the curve and AC, AC', their respective *abscisses*.

3. The most simple of all curves is the *circle*. Next to it comes a very remarkable class of curves, known by the name of *Conic Sections*, because they are figures cut out of a cone by planes.

4. Let us conceive a right cone with a circular base (Des. Geom. 85), and a plane drawn through its vertex.

Fig. 2. If this plane VD' makes with the base an angle less than that of the conical elements, then the two opposite cones will be situated, the one above, and the other below, this plane, which intersects all the elements of the surface, since they all meet at the vertex. Any cutting plane AB, parallel to it, will, therefore, intersect all the elements, on its side of the first plane, and consequently of the vertex. The curve, in that case, returns into itself, and is called an *ellipse*, (see fig. 3). This section becomes a *circle* when the plane is parallel to the base.

5. If the plane drawn through the vertex makes an angle equal to that of the elements, it will contain one of them VC and be

tangent to the cone: the cutting plane *AB'*, parallel to it, will, as in the preceding case, cut only one of the opposite cones; but, the element of contact *VC*, to which, of course, it is parallel, being intersected at an infinite distance, the curve will be a kind of infinite ellipse. It is called a *Parabola*. (See fig. 5).

6. Lastly, the plane may, like *VD*, make a greater angle than that of the generatrix: it will then cut out of the cone two elements; and, since it intersects both opposite cones, a plane *AB''* parallel to it will also intersect them; consequently the curve will have two distinct *branches;* and, because the two elements contained in the first plane *VD*, cannot be intersected in either of the opposite cones, these two branches will be infinite. The curve is called a *Hyperbola*. (fig. 4).

7. The two elements parallel to its plane, are parallel to two remarkable lines called the *asymptotes of the Hyperbola*, which continually approach to it, without ever meeting it, although they may come within an evanescent distance from it.

8. Let *MVN*, (figs. 3, 4, 5), be a vertical plane passing through the axis of the cone, and *IGAB* a section made by a plane perpendicular to that vertical plane. It is evident that the line *BA* will divide the curve into two equal parts, since the meridian plane *MVN* divides the cone in two perfectly symmetrical parts. (See D. G. fig. 19).

This line *BA* is called the *axis*, or *transverse diameter*, of the conic section.

9. The points *A* and *B*, where this axis intersects the cone, and consequently the curve, are its two *vertices*.

The parabola has but one vertex (5).

10. The perpendiculars *FG*, *HI*, to *AB* are two *ordinates* to the axis.

11. *All parallel sections of the cone are similar;* for, the corresponding abscisses *ah*, *AH*, and *hb*, *HB*, as also the ordinates *hi*, *HI*, are to each other as *Vh* to *VH*.

DEMONSTRATIONS OF SOME FUNDAMENTAL PROPERTIES OF CONIC SECTIONS.

THEOREM I.

12. *In the Ellipse and Hyperbola the squares of the ordinates are to each other as the rectangles of their abscisses.**

In the parabola these squares are as the abscisses.

Figs. 3 and 4. Through the ordinates FG, HI, pass two horizontal planes, which will cut out of the cone two circles KGL, MIN. Then, by similar triangles,

$$AF : AH :: FL : HN \text{ and } FB : HB :: KF : MH,$$

whence, by multiplication, $AF.FB : AH.HB :: FL.KF : HN.MH$. but, because FG and HI are ordinates to the above circles, as well as to the curve, we have, by a known property of the circle,

$$FG^2 = FL.KF \text{ and } HI^2 = MH.HN;$$

whence, by substitution, $AF.FB : AH.HB :: FG^2 : HI^2$. Q.E.D.

The corresponding property of the parabola might be deduced from this, by remarking that one of the abscisses of each ordinate is infinite, and becomes a common factor of the two first terms of the last proportion: but it will be more conclusive to demonstrate it as follows:

(Fig. 5.) We have, as above, $FG^2 = FL.KF$ and $HI^2 = MH.HN$: but $FK = MH$, since AH is parallel to VM; whence:

$$FG^2 : HI^2 :: FL : HN :: FA : AH.$$

THEOREM II.

13. *The horizontal projection of the section made in a right cone by a plane is also a conic section.*

[Fig. 6, 7, and 8, represent the projections of the lines of which fig. 3, 4, and 5, are the perspective.]

* For the sake of brevity, the distances of the foot of an ordinate from the two vertices, are often called the abscisses of that ordinate.

Fig. 6. *mln* being the base of the cone, *V*, *v*, its vertex, and *DPp* the cutting plane; let us conceive the horizontal projection of the curve to have been constructed, according to the methods taught in Descriptive Geometry.

Now the ordinates at the points *F* and *H* of the vertical projection, being horizontal, are equal in length to their horizontal projections *fg* and *hi*.

Besides, the lines *AF*, *FB*, and *AH*, *HB*, being in the same proportion as their projections *af*, *fb*, *ah*, *hb*, their products must also be in the same ratio, as those of the latter lines: that is;

$$AF.FB : AH.HB :: af.fb : ah.hb :: fg^2 : hi^2.$$

The projection of the curve has then the property that, the squares of its ordinates are proportional to the rectangles of their abscisses.

Consequently this projection is likewise a conic section.

14. *Corollary. In the ellipse and hyperbola, the ordinates, equally distant from the vertices, are equal;* since their squares are proportional to the rectangles of their abscisses, which are equal to each other.

15. *2d Cor. The middle C of the axis bisects any line drawn through it and limited by the curve.* For, if through the extremities *g* and *I*, of two equal ordinates, a line *gI* be drawn, it must bisect *fh*, and therefore pass through the *centre C;* since $cf = ch$ (14).

16. Any line drawn through the *centre C*, and limited by the curve is a *diameter* of it: that, which is perpendicular to the transverse axis, is distinguished by the name of *conjugate axis*. It divides, together with the transverse axis, the curve into four equal parts; and every diameter divides it symmetrically.

17. The centre *v* of the base of the cone, which is, at the same time, the projection of its vertex, is the *focus* of the *projected curve.*

18. The double ordinate *gG*, drawn through the focus, is the *parameter* of the curve.

19. The point *v'*, situated in the axis, at a distance from the other vertex *b* equal to that of the focus *v* from the vertex *a*, is a *second focus* to the same curve.

In the circle the two *foci* coincide at the centre.

20. The parabola may be considered as an ellipse with an infinite axis; for, by inclining more and more the cutting plane, the

elliptical section will approach nearer and nearer to a parabola. Consequently, the second focus, the centre and the second vertex of a parabola are at an infinite distance; or, in other words, the curve has no centre, and but one vertex and focus.

21. A diameter of an ellipse, being a line drawn from a point of the curve through its centre, will therefore be, in the parabola, a line parallel to its axis.

22. *A diameter of the parabola intersects it but in one point.* For, this diameter must cut the parabola at the same point or points where it pierces the cone which contains the curve. But the points of intersection of a line and a cone are found (D. G 158), by drawing, through the vertex, a line parallel to the given one, and passing a plane through both; this plane cuts out of the cone two elements, whose intersections with the line are the points required. In the present instance, the line drawn through the vertex is evidently the element of the cone which is parallel to the axis, and therefore to all the diameters of the parabola; consequently, this very line will be one of the two elements cut out of the cone, by the plane drawn through it and the diameter; and, since it will not meet the diameter, it follows that this latter will intersect the cone, and, of course, the curve only in one point.

23. If the secant plane be produced until it intersects the horizontal plane VD drawn through the vertex, the line of intersection will be a perpendicular to the vertical plane, on which its projection is the point D: its horizontal projection is the perpendicular Ed to the axis bad of the curve. This line will be called the *conical directrix* of the section; its projection Ed will be distinguished from it by the name of *plane directrix*, or, merely *directrix* of the projection of the curve.

In the ellipse and hyperbola, on account of the perfect similarity of the two parts of the curve situated on each side of the conjugate axis (14. 16), there is on the other side of the curve another line $E'd$ which is the directrix of the second focus.*

This directrix is at an infinite distance in the parabola.

* If an inverted cone $BV''A$ equal to the first (fig. 6, 7) be made to pass through the curve projected in AB, the projection V' of its vertex will be the second focus, and the intersection of PD and $V'D'$ will give, as in the other cone, the directrix for that focus.

THEOREM III.

The line drawn from any point of a conic section, perpendicular to its directrix, is in a constant ratio with the line drawn from the same point to the focus.

24. 1st *Demonstration.* We have by similar triangles (fig. 6, 7 & 8).

$$vg : vl :: VF : VL :: DK : DE :: FK : PE.$$

But $FK{=}gk$, and $PE{=}pd$, whence $vg : vl :: gk : pd$, or by transposition, $vg : gk :: vl : pd$. But vl is the radius of the base; pd is another constant quantity; consequently the ratio of vg to gk is also constant.

25. 2d *Demonstration.* The real length of the element FV is VG (*D. G.* 119); and we have by similar triangles:

FM or $VG : FD :: VA : AD$; that is, since VA and AD are constant quantities: *The distance of a point of a conic section from the vertex of the cone, is to its perpendicular distance from the conical directrix in a constant ratio.*

And, since all the elements make equal angles with the horizontal plane, they bear a constant ratio to their respective projections: likewise, the lines drawn from the various points of the curve perpendicular to the conical directrix, being parallel lines, are in a constant ratio to their projections; consequently,

$VG : VA :: vg : va :: FD : AD :: gk : ad$, or else, $vg : gk :: va : ad$.

26. In the ellipse, the distance to the focus is smaller than that to the directrix; for (fig. 6) VA is smaller than AD.

In the hyperbola, the former is greater than the latter, since (fig. 7) VA is greater than AD.

In the parabola, the ratio is that of equality; for $VA{=}AD$ (fig. 8).

27. 1st *Cor.* The ratio of the above-mentioned lines is that of the distance between the two directrices to the transverse axis: for, (25) $vg : gk :: va : ad :: vb : bd{=}ad'$, whence, for the ellipse, $vb+va : ad+ad' :: vg : gk$, or $ab : dd' :: vg : gk$. (fig. 6).

For the hyperbola, (fig. 7) it will be: $vb-va : ad'-ad :: vg : gk$ or $ab : dd' :: vg : gk$.

28. 2d *Cor.* *This same ratio is also that of the transverse axis to*

the distance between the foci. For $va : ad :: v'a : ad' = bd$, whence, in the ellipse, $v'a - va : bd - ad :: va : ad$, or $vv' : ab :: va : ad$.

In the hyperbola, the sum, instead of the difference of the antecedents, and of the consequents, being taken, the same result would be obtained.

29. 3d *Cor. The transverse axis is a mean proportional between the distance of the foci and that of the directrices;* for we have, $ab : dd' :: vg : gk :: va : ad$ (27) and $vv' : ab :: va : ad$ (28), whence, $vv' : ab :: ab : dd'$.

From this proposition, it will be easy to find either of them, when the two others are known.

THEOREM IV.

30. *The sum in the ellipse, and the difference in the hyperbola, of the two lines, drawn from a point of the curve to the foci, is equal to the transverse axis.*

From $vg : gk :: v'g : gk'$ (24) we obtain, for the ellipse,
$vg + v'g : gk + gk' = dd' :: vg : gk$, and, for the hyperbola,
$v'g - vg : gk' - gk = dd' :: vg : gk$; and, as in both curves, $vg : gk :: ab : dd'$ (27), we would get by substitution $vg + v'g = ab$ in the former, and $v'g - vg = ab$ in the latter curve.*

31, 1st *Cor. The line vt, drawn from the focus of an ellipse to the extremity of the conjugate, is equal to the semi-transverse.*
For, $ve = v'e$, whence $ve + v'e = 2ve = ab$.

32. 2d *Cor. In the same curve the square of the distance between the focus and centre is equal to the difference of the squares of the semi-axes.* That is, in the right angled triangle evc,

$$vc^2 = ve^2 - ce^2 = ca^2 - ce^2 \ (31).$$

THEOREM V.

33. *The tangent to a point of a conic section makes equal angles with the two lines drawn from that same point to the foci.*

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* From this article, or from article 25, it could easily be proved that, the sum in the ellipse, and the difference in the hyperbola, of the two lines, drawn from the vertex of the cone to the two points projected in H, i, and F, g, is equal to the constant quantity $VB + VA$, in the ellipse, and $VB - VA$, in the hyperbola.

1st Demonstration. Any tangent to the curve is known to be the intersection of the tangent and cutting planes.

But the trace of the tangent plane at the point g (fig. 6 & 7) is a tangent lT to the base of the cone (*D. G.* 78), that is to say, a perpendicular to the radius vl of the circle. To this trace lT, the intersection of the same tangent plane with the horizontal plane VD of the vertex must evidently be a parallel vo; and, since the intersection of the cutting plane with the same horizontal plane VD, is the directrix Ed (23); it follows that the point o, common to Ed and vo, belongs to the tangent: which is therefore the line og.

In like manner, the intersection of a perpendicular $v'o'$, to the line $v'g$, with the second directrix $E'd'$ would be another point o' of the same tangent. The points o, g, o', are consequently in the same right line; and the triangles okg, $o'k'g$, are similar; whence,

$$gk : go :: gk' : go'; \quad \text{but} \quad vg : gk :: v'g : gk' \ (24),$$

$$\text{and, by multiplication,} \quad vg : go :: v'g \ go';$$

consequently, the two right angled triangles ovg, $o'v'g$, are similar, wherefore the angles vgo, $v'go'$, are equal.

34. For the parabola (fig. 8) it will suffice to remark, that the right angled triangles ovg, okg, are equal; since go is common, and $vg=gk$ (26); whence the equality of the angles ogv, ogk.

35. *cor.* A tangent to a conic section is obtained by erecting, from the focus v, a perpendicular vo to the line, vg, which joins this point and that of contact g; and then drawing from the point o, where that perpendicular intersects the directrix, and through the point of contact, a line og, which is the tangent required.

36. From this very simple construction it results; that, *the tangent at the extremity of the ordinate of the focus is the line* du, *drawn from the point* d, *where the directrix and axis intersect.*

37. *2d Demonstration.** (fig. 9.) If it be required to draw two tangents to the curve, at the points G and g, find first the traces LT, lT, of the two tangent planes to the cone (33). The line vT, drawn from the point T, where those traces meet, to the projection v of the vertex of the cone, is evidently the projected intersection of the two tangent planes.

* The fig. shows only the base of the cone and the projections made thereon. For its vertical projection look to fig. 6.

But the intersections of the plane of the curve with the two tangent planes will be the two tangents required (*D. G.* 107), which must consequently meet in the line vT, common to those planes.

And, because, whatever the position of the points G and g may be, the triangles vTl, vTL, are equal, the angles TvL and Tvl must also be equal; whence the following general proposition:

37. *The tangents to two points of a conic section intersect in a line, which passes through the focus, and bisects there the angle made by the two lines drawn from the points of contact.*

38. This understood; if the two points G and g be the extremities of a diameter, the two tangents must be parallel, since the diameter bisects the curve symmetrically (16); consequently the intersection vT of the two tangent planes, which pass through these parallel tangents, must be parallel to them, and, because the line vT makes equal angles with vG and vg (37), it follows that either of the tangents also makes equal angles with them, or, what is the same thing, with $v'g$, and vg, since $v'g$ and vG are parallel.

39. Although the figure represents an ellipse, the demonstration may be applied to any other conic section, and even to the parabola, which may be considered as an ellipse, the second focus of which is at an infinite distance.

PROBLEM I.

To draw a tangent to a conic section when the point of contact is given.

40. 1*st Construction.* When the directrix is known, or may be easily determined,* (29, 36, 101, 133,) proceed as in No. 35.

41. 2*d Construction.* Draw, from the foci to the point of contact, the lines vg and $v'g$; (fig. 9) produce one of them, $v'g$, of a quantity gw equal to the other, and draw, from the point g to the line vw, a perpendicular gh, which will be tangent to the point g (33): for, the triangles gwh, gvh, are equal, and consequently also the angles wgh and hgv.

* In the ellipse, the directrix will be readily determined by drawing through the focus, and to the line $v'e$, a perpendicular, and finding the point where it intersects the tangent at the extremity of the conjugate axis, a tangent which is evidently parallel to the transverse.

42. Fig. 10. In the hyperbola, instead of being added to $v'g$ the line $gw = gv$ must be subtracted from it.

43. Fig. 11. In the parabola the line gv' drawn to the second focus is, of course, parallel to the axis. The construction is, otherwise, the same as for the ellipse.

PROBLEM II.

To draw a tangent to a conic section from a point given without the curve.

44. From the given point A, with a radius Av equal to its distance from either focus, describe a circle; and then, from the other focus, with a radius equal to the axis, describe a second circle, whose intersection with the first will give the point w, and consequently the line vw, to which draw the perpendicular Ahg, which is the tangent: for $Av = Aw$, whence $vh = wh$; besides $v'w$ is equal to the axis.

If the point of contact were required, the line $v'gw$ would determine it.

The second point of intersection of the two circles would give a second tangent Ag' to the curve. (fig. 9).

45. It is evident that the circle described from the focus v', with a radius equal to the transverse, must intersect this axis in a point situated at a distance from the vertex, equal to the distance between the focus and vertex; that is, $bv' = av = am$; but, in the parabola, (fig. 11) the circle described from the second focus, having an infinite radius, becomes a straight line, and this line is the directrix itself, since $va = ad$ (26): its intersection with the first circle gives, then, the point w as above.

This, and other propositions relative to the parabola, show sufficiently that its properties are deducible from those of the ellipse and hyperbola: they will, however, be demonstrated in a more direct way, when we shall treat of the parabola.

46 1st *Cor. If a line be drawn from either focus perpendicular to a tangent, the distance of their intersection from the centre is equal to the semi-transverse axis.*

For, vc and vh are respectively the half of vv' and vw; consequently ch is one half of $v'w$ to which it is parallel.

47. 2*d Cor. A circle described on the transverse axis will pass*

through the point h *where a line drawn from the focus perpendicular to the tangent intersects it.*

48. In the parabola, the radius of this circle being infinite, the circle itself becomes *the tangent* ah *at the vertex, which must consequently pass through the point of intersection of the tangent and its perpendicular.*

49. *3d Cor.* From this we deduce the following construction of the tangent: Describe a semi-circle on the transverse axis; draw a parallel *ch* to the line *v'g*, and join the point *h* where it cuts the circle to the point of contact.

50. *4th Cor.* The foci might be obtained from the tangent by means of the perpendicular *h'v'* and *hv* to that line.

51. *5th Cor.* If a line *cP* be drawn through the centre parallel to the tangent, the line *go* will evidently be equal to *ch*, and consequently to the semi-transverse (46). The line *cP* being called the *conjugate diameter* of *gc*, as will be seen hereafter, the proposition may be expressed thus: *If a line be drawn, from the extremity of any diameter, to either focus, the part of it intercepted between this diameter and its conjugate is equal to the semi-transverse.*

This property belongs evidently only to the ellipse and hyperbola.

52. These few fundamental propositions will suffice to give a correct idea of the nature of conic sections in general. We will now treat of each individual curve, and present those properties that are peculiar to each one, or that are more easily obtained from some particular position of the curve. We will afterwards resume the general mode of demonstration, for some useful propositions common to them all.

OF THE ELLIPSE.

53. Many properties of this curve may be obtained immediately, either by supposing it to be projected into a circle, (fig. 13), which is the case when it is cut out of a right cylinder. *(D. G.)*

54. Or, by considering it as the projection of a circle *AE'B* (fig. 12), which has been raised, round its diameter *AB*, to an oblique position *ce* (13).

55. In both cases the altitude *ef*, of the extremity of the inclined diameter *ce* is equal to the distance between the focus and centre;

for it is the second leg of a right angled triangle, of which the hypothenuse is one half of the transverse axis, and the other leg its semi-conjugate (31).

56. Hence the following construction to find the foci: Describe on the axis a semi-circle $AE'B$, and, through the extremity E of the conjugate, draw a parallel EF' to the axis; the points f', F', where it cuts the semi-circumference, being projected upon the axis in the points f and F, these latter will be the foci.

THEOREM I.

57. *If a circle be described on either the transverse or the conjugate axis of an ellipse, an ordinate in the circle, will be to the corresponding ordinate in the ellipse, as the axis of this ordinate, is to the other axis.*

Fig. 12 and 13. GH being the horizontal projection of the ordinate $cg' = HG'$, we have by similar triangles, $ce : cf :: cg' : cg$; or, by substituting the lines equal to them, $CE' : CE :: HG' : HG$; but $CE' = AB$, and $2CE = EL$; consequently also,
$AB : EL :: HG' : HG$. Q.E.D.

58. *Cor.* Because $G'H^2 = AH.HB$, the preceding proportion will become, by squaring its terms, $AB^2 : EL^2 :: AH.HB : HG^2$; a proportion which answers for either axis and which, in respect to the transverse axis, may be expressed thus:

As the square of the transverse, is to the square of its conjugate, so is the rectangle of the abscisses, to the square of their ordinate.

THEOREM II.

59. *The parameter is a third proportional to the transverse and the conjugate.*

Fig. 12. F being the focus, $2FP = P$ (the parameter), by definition, and we have (57):

$CE' : CE :: FF' = CE : FP$, whence, because, $2CE' = AB$, and

$$2CE = EL, \qquad 2FP = P = \frac{EL^2}{AB}.$$

60. 1*st Cor. The semi-conjugate is a mean proportional between the distances of either focus from the vertices;*

that is, $F'F^2 = CE^2 = FA.FB$.

61. *2d Cor. As the transverse is to its parameter, so is the rectangle of the abscisses, to the square of their ordinate.* For (58) $AB^2 : EL^2 :: AH.HB : HG^2$, whence, because, $EL^2 = P.AB$ (59), $AB : P :: AH.HB : HG^2$.

THEOREM III.

62. *The tangents at the two corresponding points of an ellipse and a circle described on either the transverse, or the conjugate, axis intersect in that common axis.*

For, one of them is the base of a cylinder, of which the other is a section. Let us take for example the case of fig 12. AEB being the projection of the circle $AE'B$, when raised to the position *ce*, the tangent IG to the former is the trace of the tangent plane to the projecting cylinder. The point I, where this trace cuts the trace AB of the plane of the circle, belongs to the tangent to this latter (*D. G.* 109). But, when the plane of the circle is turned down, round the fixed axis AB, the point I remains stationary; and as the point G,g', must fall in the ordinate HG produced, it follows that IG' is the tangent to the circle.

63. *Cor.* This furnishes another very simple and obvious construction of the tangent to an ellipse, when the point of contact is given.

64. But, *if the tangent were to be drawn from a point* m' *given without the curve*, it would be done by finding a fourth proportional Mm, to the conjugate, the transverse and the ordinate $m'M$ of the given point; and, then, drawing, through the point m thus obtained, a tangent to the circle described on the transverse axis: the point of contact G' would show the ordinate of the corresponding point in the ellipse.

65. *2d Cor. The semi-transverse is a mean proportional between IC and HC.* For, the radius CG' of the circle is equal to the semi-transverse, and $CG'^2 = CA^2 = CH.CI$. *The like proposition obtains for the semi-conjugate* (fig. 13).

66. *In general, all such properties as relate to the parts of, either the conjugate or the transverse axis, are the same as in the circle:* Since these two axes and their parts remain, either the same, or proportional, in the projection (fig. 12 and 13.)

67. For instance, $CH : CI :: AH^2 : AI^2$* in the circle, and consequently in the ellipse (fig. 12 and 13.)

68. In like manner, $CH.HI = AH.HB$; since, in the circle, they are equal separately to $G'H^2$.

69. Again, $IA : IH :: IC : IB$, or $IA.IB = IC.IH$; for, in the circle, these products are equal to $G'I^2$.

THEOREM IV.

70. *If, from any point in the curve, there be drawn an ordinate and a normal to the curve, or perpendicular to the tangent; then,* (fig. 12)

The subnormal DH, is to the distance between the centre and ordinate (or subnormal in the circle) *as the square of the conjugate is to the square of the transverse axis:* that is,

$$DH : CH :: CE^2 : CA^2.$$

For (57) $EC^2 : E'C^2 :: GH^2 : G'H^2$; and, because, in the right angled triangle $CG'I$, $G'H^2 = CH.HI$, and in the triangle IGD, $GH^2 = DH.HI$, the above proportion will be obtained by substitution.

THEOREM V.

71. *The areas of all such figures, as are the projections of corresponding figures in the circle, preserve in the ellipse the same ratio they had in the circle.*

This is an evident consequence of the following well known proposition:

72. *The area of any figure situated on an inclined plane, is to its projection, as the slant of the plane, to its own projection. Or, as radius, to the cosine of the angle of inclination of the plane.*

Fig. 14. The triangle BaC, is to the triangle BAC, of which it is the projection, as the altitude aP of the former, is to the altitude AP of the latter; that is to say, in the above mentioned ratio.

The same thing is true of the two triangles BdC, and BDC:

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* The angles $JG'A$ and $AG'I$ (fig. 13) are equal, whence $HA^2 : IA^2 :: G'H^2 : G'I^2 :: CH.HI : HI.CI :: CH : CI$.

since they also stand on the same base ; and, consequently, it will hold for their respective differences *Cda*, *CDA*, with the two former triangles *BaC*, *BAC*.

73. This proposition obtaining for a triangle, will hold good for any polygon, or plane figure whatever.

74. *Cor.* *All the parallelograms circumscribed about an ellipse are equal to one another, and are each equal to the rectangle of the two axes.* For, they are the projections of squares tangent to the circle, of which the ellipse is the projection (fig. 15).*

Definitions.

Fig. 15. 75. If two diameters be drawn parallel to the sides of a square circumscribed about a circle, they will each pass through the points of contact of the circle and the sides to which they are perpendicular. That is, *GH* passes through the points *G* and *H*, where *AB* and *DC* touch the circle.

If now the figure thus formed be placed on an inclined plane, and then projected upon a horizontal one, the right angles will, in general, become oblique, and, the projections of parallel lines being parallel, the projections of the two perpendicular diameters *GH*, *EF*, will be two parallels *gh*, *ef*, to the sides of the parallelograms *abcd*, into which the square *ABCD* is projected ;† and these two lines *gh*, *ef*, will pass through the points of contact *g*, *h* and *e*, *f*.

76. The projections of two perpendicular diameters of the circle are two *conjugate diameters* of the ellipse.

77. The ordinates, like *PN*, which, in the circle, are perpendicular to its diameter *GH*, and parallel to the other diameter *EF*, will be projected in other ordinates, like *pn*, oblique to the projection *gh* of the former diameter, and parallel to that *ef* of the latter.

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* This is to be understood of such parallelograms only as are the projections of squares circumscribed about the circle, that is to say, of parallelograms that have their sides parallel to two conjugate diameters (76).

† The circle is supposed in fig. 15. to have been turned down upon the horizontal plane; the corresponding points of the two figures will accordingly be found in the same ordinates *Gg*, *Bb*, &c. to their common axis or diameter.

78. Such ordinates are evidently parallel to the tangent at the extremity of the diameter, to which they are applied.

79. *Cor.* Any diameter of the ellipse being given, its conjugate will be found, by drawing it parallel to the tangents at the extremities of the former.

THEOREM VI.

80. *If there be two tangents drawn to the extremity of any two diameters; each tangent meeting the other's diameter produced, the two triangles so formed will be equal.* That is, $lIO = MgO$.

For, they are the projections of two equal triangles, LIO, MGO, (72). They are also equal to the triangle nmO; since the original triangle $NmO = LIO$.

THEOREM VII.

81. *Any diameter bisects all its double ordinates, or parallels to its conjugate.*

For $n'n$ is the projection of the chord $N'N$ of the circle, which is bisected by the diameter GH, of which gh is the projection.

THEOREM VIII.

82. *The squares of the ordinates to any diameter are to each other, as the rectangles of their respective abscisses.*

For, in the circle, the squares of the ordinates are equal to the rectangles of their abscisses, and consequently in the ratio of these rectangles. But the ratio of the ordinates must be the same when projected, since they are parallel lines: and the abscisses, being parts of the same diameter, must, in the projection, preserve their original proportion: wherefore the theorem is manifest, not only for the axis (12), but for any two conjugate diameters.

83. In general, all such propositions, as relate to the axes, hold good for any pair of conjugate diameters: for, parallel lines, as well as the different portions of the same line, will, when projected, preserve their original ratio, tangents will continue so, the points of intersections of lines will correspond, &c. &c.

Hence, the propositions of Arts. 58, 66, 67, 68, 69, must be extended to conjugate diameters and their oblique ordinates.

THEOREM IX.

84. *The sum of the squares of two ordinates, applied to the transverse axis from the extremities of two conjugate diameters, is a constant quantity equal to the square of the semi-conjugate axis.* That is, (fig. 17) $eg^2+fh^2=CD^2$.

For, the small circle, described on the conjugate axis, being considered as the projection of the ellipse, and EC, CF being two perpendicular radii, Ce and Cf will be two conjugate diameters of the ellipse (76, 77).

Now, the triangles FHC, GEC, are equal, since $CF=CE$, and their sides are perpendicular; consequently, $FH=GC$; which, being substituted in the equality, $FH^2+HC^2=FC^2=CD^2$, gives $CD^2=GC^2+HC^2=eg^2+fh^2$.

85. Fig. 16. *Shows that the same property obtains for the abscisses* GC and CH, *which are ordinates to the conjugate axis*: that is, $GC^2+CH^2=eg^2+fh^2=AC^2$.

THEOREM X.

86. *The sum of the squares of two conjugate diameters is equal to the sum of the squares of the transverse and conjugate axes.*

We have (fig. 16.) $Ce^2=CG^2+Ge^2$, and $Cf^2=CH^2+Hf^2$; and, because $Ge^2+Hf^2=Cd^2$(84); and $GC^2+HC^2=AC^2$(85), it follows that $Ce^2+Cf^2=Cd^2+AC^2$.

AREA OF THE ELLIPSE.

87. *The area of the ellipse is a mean proportional between the inscribed and circumscribed circles.*

Fig. 16. For, if we consider the ellipse as the projection of the circumscribed circle (54), we have, by article 72 (calling C the circumscribed, c the inscribed circle, and E the ellipse).

$E : C :: Cd : CD=AC$; again (53)

$E : c :: CA : CI=Cd$; whence, by multiplication, $E^2=Cc$.

OF THE HYPERBOLA.

88. Of the different hyperbolas, which may be cut out of a cone, the most convenient to demonstrate the properties of that family of curves are those, whose planes are parallel to the axis of the cone. (fig. 18, 19.)

Fig. 19. represents a hyperbola projected upon the meridian plane, to which it is parallel: *fe* is the horizontal projection of the plane of the curve; *C* is the vertex of the cone, and *Ofe* its base.

89. The lines *CO*, *CP*, since they are in the vertical meridian plane, will be parallel to the secant plane *fe*, and consequently the point of the curve, situated in each of them, will be at an infinite distance (6).

If, now, two tangent planes were drawn along the elements *CO*, *CP*, they would intersect the plane of the curve in two tangents to the points situated in those elements, *CO*, *CP* (33); and, since these points of contact are at an infinite distance, the tangents will be parallel to the elements themselves, and approach nearer and nearer to the curve, without ever meeting it. But the two tangent planes, along the elements *CO* and *CP*, are perpendicular to the vertical plane which contains them; therefore these lines are also the projections of the two infinite tangents, or, in other words, *the Asymptotes* of the hyperbola.

It is otherwise easy to understand that the curve will approach nearer and nearer to its asymptotes, within any assignable distance; for any element, however little it may differ from the asymptotic elements *CO*, *CP*, will be oblique to the cutting plane, and will, of course, intersect it in a point of the curve: consequently, there is no limit to the nearness of the hyperbola to its asymptotes.

90. Again; however near a hyperbola may be to its asymptotes, it is always possible to conceive between them an infinite number of other hyperbolas that will intersect neither.

For, between the plane *fe* of the curve, and the meridian vertical plane *OCP*, there may be drawn an infinite number of planes, like *f'e'*, parallel to both. Each of these planes will cut out of the cone a hyperbola, like that which is dotted in the figure, and the projections of these new hyperbolas will all be included between

their common asymptotes CO, CP, and the first curve; since, for any horizontal section, the ordinate DE' or de' of the intermediate hyperbola will evidently be greater than DE and smaller than DP.

91. The same mode of reasoning would hold for any hyperbola other than a vertical one; but, in order to apply it, it is necessary to know how to construct the asymptotes in the general case.

Fig. 6. exhibits the operation: the plane $VRrr'$: parallel to that of the curve, cuts the cone in the elements rv, $r'v$; and the traces rx, $r'x'$ of the tangent planes along these elements intersect the trace Ppx' of the plane of the curve in the points x and x', through which the infinite tangents, or asymptotes, must pass. The parallels xC, $x'C$, to the elements rv, $r'v$, are consequently the asymptotes (7, 89).

92. Any line MN, passing through the centre, and terminated by the hyperbola, is a *diameter*.

And its *conjugate* is the line il, parallel to the tangent IL, at the extremity of the diameter, and equal in length to the part IL intercepted by the asymptotes.

93. The perpendicular Ca to the transverse is the conjugate axis. It is evidently equal to the distance of the cutting from the meridian plane. For AR is the radius of a circle, which passes through the vertex A of the curve.

94. When the asymptotes are at right angles, the *hyperbola* is called *equitateral*. Its axes are equal to each other, as also any two conjugate diameters.

95. Two hyperbolas that have the same asymptotes and axes; so, however, that the conjugate of one is the transverse of the other, are called *conjugate hyperbolas* (fig. 19). They may be cut out of the same hyperboloid by two parallel planes, one within and the other without the small circle. (*D. G.* fig. 23. 24. Arts. 152, 153).

96. The analogy between the hyperbola and ellipse is such, that almost all the properties that belong to one attend the other; as will be readily perceived in the following propositions.

THEOREM I.

97. *As the square of the transverse, is to the square of its conjugate;*

So is the rectangle of the abscisses, to the square of their ordinate.

In the circle HEG (fig. 18*), we have $HD.DG=DE^2$; but, by similar triangles, aA or $aB : aC :: AD : DG$, and
$aB : aC :: DB : DH$, whence,
by multiplication, $aB^2 : aC^2 :: AD.DB : DG.DH=DE^2$.

THEOREM II.

98. *The square of the focal distance, which, in the ellipse, is equal to the difference, is, in the hyperbola, equal to the sum, of the squares of the semi-axes.*

Fig. 20. We know (47), that a circle described, on the centre C, with a radius equal to the semi-transverse axis, will pass through the point of intersection of any tangent, and the perpendicular let fall upon it from the focus.

But the asymptote CS is a tangent to the curve (89); consequently, the circle must pass through the foot P of the perpendicular FP; and CP is equal to the semi-transverse CA.

99. The triangles ACB, FCP, are therefore equal; whence $FP=AB=$ the semi-conjugate (93); $FC=CB=$ the focal distance; and $BC^2=FC^2=AC^2+AB^2$.

100. 1*st Cor.* An arc, BF, described from the centre C with the radius CB, will show the focus; consequently, *the semi-conjugate is a mean proportional between the distances of either focus from the two vertices*; that is, $AB^2=AF.Af=AF.FG$.

101. 2*d Cor.* Because the semi-transverse is a mean proportional to the focal distance, and the distance between the centre and directrix (29), this latter will be the perpendicular PD let fall, from the point P, upon the axis; for $AC^2=PC^2=CD.CF$.

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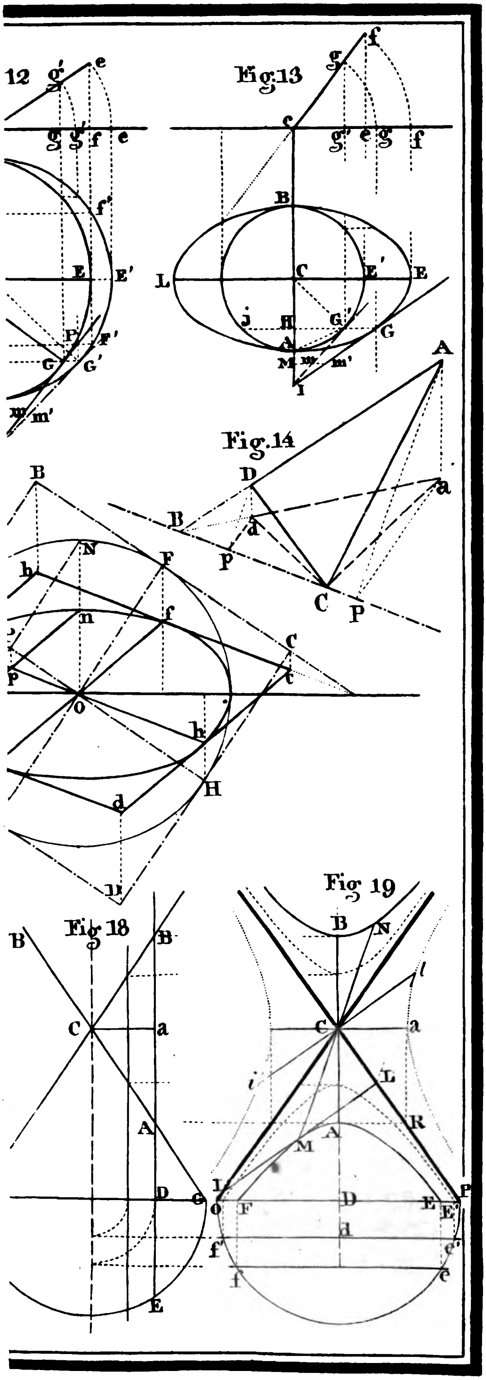
* In this figure the plane BDE of the curve is perpendicular to the vertical plane.

Plate 1.



THEOREM III.

102. *The parameter is a third proportional to the transverse and its conjugate.*

The tangent at the extremity E of the focal ordinate passes through the point D of the directrix and axis (36); consequently, Ed or $FD : FE :: AC : CF$ (24. 28), whence

$$FE.AC = CF.FD = FP^2 = AB^2 \text{ (99), and } FE = \frac{AB^2}{AC}.$$

103. *Cor.* The same proposition as in art. 63 may be proved for the hyperbola, by combining, as in that article, the last found equation with the proportion, $AC^2 : AB^2 :: FA.FG : FE^2$ (97). the result would be, $AC : FE = \frac{1}{2} P :: FA.FG : FE^2$.

THEOREM IV.

104. *If a tangent and ordinate be drawn from any point in the curve, the semi-transverse is a mean proportional between the distances of the centre to the ordinate, and of the same point to the intersection of the tangent and axis.* That is, $AC^2 = CD.CT$.

Fig. 21. CPQ being the vertical projection of the cone, PMQ its base, and MN the horizontal trace, or projection, of the vertical plane which cuts out the hyperbola projected in EAe; if it be required to find the tangent to the point E, draw to the cone a tangent plane, the horizontal trace of which will be the line FMI, tangent to the base (D. G. 78); and its vertical trace the line FC (since the vertex C, which is a point of the tangent plane, is in the vertical plane of projections).

105. To that trace CF the tangent ET must be parallel, since it is the intersection of the tangent plane, and a plane parallel to that of projections.

Now, construct the horizontal circle cut out by the plane lA, which passes through the vertex A of the hyperbola; this small circle is projected on the horizontal plane in bma; its tangent Lg, at the point m of the element of contact MmD, is in the tangent plane, and must consequently pierce the vertical plane in a point of its trace FC. This point is found at l by erecting the perpendicular Ll.

106. The figure being well understood, we have, $gh=aD=mD$, and consequently the triangles Lgh, LmD are equal; whence $LD=Lg=FM$; but $FM^2=FE.FD$, and, by similar triangles, the quantities $lA=LD$, or FM, FE and FD, bear respectively the same ratio to CA, CT, CD, whence, by substitution, $CA^2=CD.CT$.

107. 1st *Cor.* *If a circle be described on the transverse axis, and a tangent OD drawn to it from the foot D of the ordinate of the hyperbola, the ordinate OT of the point of contact passes through the point T, where the tangent to the hyperbola intersects its transverse axis.* For, by a known property of the circle, $CO^2=CD.CT$.

108. This corollary furnishes a very evident construction of the tangent, for which it is only necessary to know the transverse and a point of the curve. E being the given point, describe on the transverse a circumference, to which draw the tangent DO; and then join the foot T of the ordinate of the tangential point, to the given point E by a line TE, which is the tangent required.

109. By erecting the perpendicular nf to the tangent, at the point where it cuts the circle, the focus may now be determined; and, from it, the conjugate axis and the asymptotes. (46, 100.)

110. 2*d Cor.* We have, in the circle, $CD : CT :: AD^2 : AT^2$. Consequently this same proposition is true for the hyperbola, as well as for the ellipse (67).

111. 3*d Cor.* Again, $TA : TD :: TC : TA'$, or $TA.TA' = TD.TC$; for, these rectangles are separately equal to OT^2. This property corresponds to that of art. 69.

THEOREM V.

112. *If, from any point in the curve, there be drawn an ordinate and a normal, then, as in the ellipse: the subnormal OM* (fig. 20), *is to the distance CM between the centre and ordinate.*

As the square of the conjugate, is to the square of the transverse.

Let us first construct the normal: the two asymptotes being still considered as including the projection of a cone, find the radius HO of a sphere inscribed in that cone (D. G. 243) and tangent to it in the circle, whose projection, on the plane of the figure, is the ordinate HIL of the point of a contact I. It is evident, that all such radii of the sphere, as pass through the circle

HIL, are normal to the cone, and that the normal, which passes through the point I, being perpendicular to the tangent plane, its projection IO, on the plane of the hyperbola, must be perpendicular to the trace of the tangent plane; which trace is the tangent IT to the hyperbola (105, D. G. 37).

Now, in the right angled triangle OHC, we have,

$HO^2 : HC^2 :: OM : MC$; and, by similar triangles,
$HO : HC :: NG : GC$; whence, by substitution,
$NG^2 : GC^2 :: OM : MC$.

THEOREM VI.

113. *If a line be drawn across a hyperbola, the two parts intercepted between each asymptote and the curve are equal.*

Fig. 22. For, if a section be made through the line AB, and perpendicular to the vertical plane, and then revolved over upon it, the ordinates CD, cd, of the section will each be equal to the distance of the hyperbola from the vertical plane: from the equality of the ordinates CD, cd, it follows, that their corresponding abscisses AD, dB, are also equal (14). In like manner, it might be proved, that $db = d'b'$.

114. 1st *Cor. The point of contact of a tangent* ab *bisects it.*

115. 2d *Cor. Any diameter bisects all its double ordinates, or parallels to its conjugate.*

For, $AE = EB$, from which subtracting the equal parts AD and dB, it remains $DE = dE$.

THEOREM VII.

116. *If any number of parallel lines be drawn between the two asymptotes, each being divided by the curve into two segments; the rectangle of these two segments is, for all the parallel secants, a constant quantity equal to the square of the semi-diameter, to which the lines are parallel.*

Fig. 22. For, the two parallel ellipses, afb, AFB, are similar (11); whence, $fe^2 : ae^2 :: FE^2 : AE^2 :: CD^2 : AD.DB$ (58); but, $fe = CD$; consequently $ae^2 = AD.DB$.

117. *Cor. When the lines are perpendicular to the axis, the rectangles of their segments are each equal to the square of the semi-conjugate.*

THEOREM VIII.

118. *If any number of lines be drawn from a given point, so as to intersect two other lines; and, if all the intercepted parts be divided in a given ratio, all the points of division will be in a hyperbola, the asymptotes of which will be parallel to the fixed lines.*

Fig. 23. From the point D, draw the lines DG, Dg, &c.; make $HG=DE$, $hg=De$, &c.; it is evident, that all such points, as, G, g, &c. are in a hyperbola, having the fixed lines, CBC, ABA, for its asymptotes (113).

119. If now the lines, DG, Dg, &c. be bisected, the points of division, P, p, &c. will be respectively the middle of the lines EH, eh, &c. But the hyperbola Ggj may be considered as the base of a cone, and D as the projection of its vertex. A horizontal plane which would bisect all the elements of that cone, would cut out of it a curve similar to its base (11), and the projection of which must be the curve BPp, which passes through the middles P, p, of the projections DG, Dg, of the conical elements:

The curve BPp, similar to Ggj, is therefore a hyperbola.

That a similar curve would be formed by dividing EH, eh, &c. in any ratio, may be as easily proved. Let us show it for the ratio of 2 to 1.

If HG, hg, &c. be bisected, the curve passing through all the points of division G', g', &c. will be another hyperbola: for, if a line $B'A'$ be drawn parallel to BA, so as to bisect DE, De, &c. then $DE'=HG'$, $De'=hg'$, &c.

Divide now DG', Dg', &c. in the ratio of 2 to 1; the points so obtained will divide in the same ratio the lines EH, eh, &c.; and, considering, as above, the curve $G'g'j$ as the base of a cone, &c. it will be seen that the points of division P', p', &c. belong to a hyperbola.

THEOREM IX.

120. *All the parallelograms are equal, which are formed, between the asymptotes and curve, by lines drawn parallel to the asymptotes.* That is, $BADC = badC$.

Fig. 24. To the point A draw the tangent EF; and, through the other point a, the parallel eaf. Then will $ea.af = EA.AF$ (116); but $AE : AB :: ae : ab$, or $AE.ab = AB.ae$, whence, by multiplication and reduction, $ab.af = AB.AE$, from which equation, it appears plain that the parallelograms $BADC$ and $badC$ are equal.

121. 1*st. Cor. All the triangles, like CEF, formed by the asymptotes and any tangent to the curve are equal to one another.*

Because $EA = AF$, the triangles EAB, DAF, CDA, CBA, are equal to each other; consequently the parallelogram $BADC$ (whose area is a constant quantity) is one half of the triangle CEF, which is, therefore, also a constant quantity.

122. 2*d Cor. All the parallelograms inscribed between four conjugate hyperbolas are equal to one another, and consequently to the rectangle of the two axes.*

For every one of them, like $GEFH$, is formed by four tangents to the conjugate hyperbolas, which are equal and parallel to the conjugate diameters gh, AI, (95); but the triangles ECF, FCH, HCG, GCE, are equal to one another; and consequently the parallelogram, which is their sum, is a constant quantity.

THEOREM X.

123. *If there be two tangents drawn, the one to the extremity of the transverse, the other to the extremity of any diameter, each tangent meeting the other's diameter produced, the two tangential triangles so formed will be equal.* That is, $CGb = Cop$.

Fig. 20. Cbp being considered as the element of a cone, if it is revolved round the axis CG, so as to come into the position $Cp'L$, then the points b and p will coincide with b' and p'; and the triangles Cop', Crp, will be equal: since the area of the former is $\frac{1}{2}Co\,(np+pp')$, that of the latter $\frac{1}{2}Co\,(rs+np)$, and $pp' = \frac{1}{2}rt = rs$ (114).

But the triangle $Crp = CGb'$ (121), consequently $Cop' = CGb'$:

and, therefore, when these two last triangles are, by a rotation round the axis CG, restored to their former position, their projections CGb, Cop, must also be equal.

THEOREM XI.

124. *The squares of the ordinates to any diameter are to each other as the rectangles of their respective abscisses.*

This, as well as all the properties that belong to the two axes, must be understood, (as in the ellipse), of any pair of conjugate diameters; since, a hyperbola being projected upon another plane, its rectangular diameters will, in general, become oblique to each other. Whence the same conclusions as have been obtained in arts. 80, 81, 82, 83, of the ellipse.*

THEOREM XII.

125. *The difference of the squares of two ordinates applied to the transverse axis, from the extremities of two conjugate diameters, is a constant quantity, equal to the square of the semi-conjugate.*

That is, $PM^2 - OH^2 \quad AB^2$ (Fig. 24. bis.)

In the first place, let us draw to the point O the tangent OD; and consider the circular section, made through the point D, as the base of the cone. The line dh being the projection of the plane of the hyperbola, d must be the point where the tangent OD pierces the plane of the base, and consequently dg will be the trace of the tangent plane; GC, gc, the element of contact, and i the horizontal projection of the point of contact O.

Now, since $DO \quad PC$, it follows that $PM = DI = di$, and, because $dg = Dd = ch$, the triangles chi, igd, are equal, and $di = ic$; whence $di^2 - ih^2 = ch^2$, or, by substitution of equal quantities,

$$PM^2 - HO^2 = AB^2.$$

126. *Cor.* it could be proved, in the same manner, and by means of the conjugate hyperbola, that, $OL^2 - PN^2 = CA^2$.

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* It is proper to remark here, that it results from the note to art. 240 (D. G.), that an acute angle may be projected in a right angle upon an infinite number of planes. This, together with the property of a hyperbola, to be projected into another hyperbola, completes the foregoing propositions.

THEOREM XIII.

127. *The difference of the squares of any two conjugate diameters is a constant quantity, equal to the difference of the squares of the axes.* For, $PC^2=PM^2+PN^2$; and $OC^2=OL^2+OH^2$; whence, by subtraction and substitutions (125, 126),

$$OC^2-PC^2=CA^2-AB^2.$$

128. From the preceding theorems, it must now appear evident, that almost all the properties of the hyperbola correspond to those of the ellipse, from which, or from the circle, they may be easily deduced, by means of sections, or of projections of the cone.

OF THE PARABOLA.

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129. All the properties of this curve might be derived as corollaries from those of the ellipse. It is, however, more simple and more direct to demonstrate them as in the following paragraphs.

THEOREM I.

130. *The abscisses are proportional to the squares of their ordinates.* (Art. 12).

THEOREM II.

131. *The distance of a point in the curve from the focus is equal to the distance of the same point from the directrix* (26).

Fig. 25. The plane GD, cutting out of the cone MVN a parabola projected in gag', we have $gv=vm=OM$; and, because of the equality of the triangles DEG, VOM, the lines OM and EG are equal; whence, EG or $kg=vg$.

132. 1st *Cor. The distance between the focus and directrix is equal to half the parameter.* That is, (18) the semi-parameter $vf=fe=vd$.

133. 2d *Cor. The distance between the focus and vertex is equal to the distance of this latter to the directrix; and both are equal to one fourth of the parameter.*

For, it results from the theorem that $va=ad$; which may otherwise be made evident, by remarking that they are the respective projections of the equal lines VA and AD.

THEOREM III.

134. *A tangent to a point in a parabola makes equal angles with the axis, and the line drawn from the point of contact to the focus.* Or, else: *this tangent bisects the angle made by the lines drawn from the point of contact, the one to the focus, and the other perpendicular to the directrix.* (See Art. 34. 43.)

135. 1*st Cor. The line, drawn from the focus to the point of contact, is equal to the distance between the focus and the point of intersection of the tangent and axis.* That is, $vg=vn$. For they are opposite to the equal angles vgn, vng.

136. 2*d Cor. The tangent passes through the point* n, *where the axis is intersected by the circle described, on the focus, with a radius equal to its distance from the point of contact.*

This furnishes a very ready construction of the tangent.

137. 3*d Cor.* Since vg $gk=dh$, then $dh=vn$.

138. 4*th Cor. The sub-tangent is double the abscisse of contact.* That is, $nh=2\ ah$; for ah and an are the projections of the two equal lines AG and AN.

139. 5*th Cor. The perpendicular drawn from the focus to the tangent bisects it.* For nvg is an isosceles triangle.

Again, this perpendicular vb produced must pass through the point k, because $kg=gv$, whence $vb=bk$.

140. 6*th Cor.* Consequently the tangent at the vertex passes through the point b; and is a mean proportional between the abscisse ha and the distance of the focus to the vertex. For, in the right angled triangle nbv, $ba^2=na.av=ah.ad$.

THEOREM IV.

141. *The subnormal is a constant quantity equal to half the parameter.**

Because the tangent gn passes through the extremity n of the diameter nm, the normal, or perpendicular, gm, must pass through the other extremity m of the same. Therefore,
$mh=MG=DV=vd=\frac{1}{2}$ parameter (132).

142. *Cor.* From this, we get; $gh^2=nh.hm=2ah.\frac{1}{2}P=P.ah.$ That is: *the square of an ordinate is equal to the rectangle of its abscisse and the parameter.*

This is also another demonstration of Theorem I.

THEOREM V.

143. *If two tangents be drawn from the same point of the axis and a third tangent intersect them; the part of this latter intercepted between its point of contact and one of the first tangents is equal to the other part of it contained between the axis and the other tangent.* That is: $GI=AH$ (Fig. 26,)

Let us suppose that the cone used in the preceding figure, has been made to turn round its axis, so as to present the parabola in front. Then, evidently, the two extreme elements DC, DE, will be tangent to the vertical projection of the curve; and DM will be the projection, both of the axis of the cone and of the element along which the tangent plane, parallel to that of the parabola, touches the cone. (This element is VM in fig. 25.)

Now, through the tangent GH, pass a plane perpendicular to the vertical plane, it will cut out of the cone an ellipse, which suppose to be revolved round its axis GH. The point I will be found in i; the point A in a; and the ordinates Ii, Aa, will be equal: for, the tangent to the ellipse, at the point I or i, is the same as the tangent to the parabola, in the plane of which it is, of course, situated; and the tangent ab to the point A or a, being the intersection of the plane of the ellipse and the tangent plane parallel to that of the curve, it follows that the tangents if and ab are contained in two parallel planes; and consequently are parallel.

But parallel tangents, in an ellipse, have their points of contact equally distant from the vertices. Whence, as above, $AH=GI$.

144. *1st Cor.* The lines GC and DH are equal; for, $BM=BD$, and $BQ=BA$ (138); and consequently their differences, QM or IN, and AD, are also equal. But the right-angled triangles GIL, PAH, have their hypothenuses and angles equal; whence $PA=IL$, and, by subtraction, LN or $GO=PD$. Hence the triangles GOL, DPH, are also equal, and $GC=DH$.

145. *2d Cor.* If the tangents DC, DE, be each divided into the same number of equal parts, and the opposite points of division joined; by as many lines GH, $G'H'$, &c. each one of these lines will be tangent to the parabola.

146. *3d Cor.* A parabola being a kind of ellipse, its projection will be another parabola: the projection of the axis DM will be a diameter of the projected curve; the tangents CD, ED, will be projected into tangents to the same, but will generally become unequal; in all cases, however, equal parts of the same line will remain equal in the projection.

147. Consequently for any three tangents whatever we shall have (fig. 27), $gi=ah$, $ga=ih$; and also, $dh=hh'=h'h''$ &c., if cg, gg', $g'g''$, be equal to each other. (145).

148. *4th Cor. If there be three tangents intersecting each other, their segments will be in the same proportion.* That is,

$$gc : gd :: gi : ih :: dh : he \text{ (fig. 27).}$$

For (fig. 26.) the equality of the triangles GOC, CPH (144) shows that $CO=PH$; and that of the triangles GIL, PAH, gives $GL=PH$; whence, $CO=GL=ON$ and $CG=GG'$; hence, GG' or $CG : GD :: GI : GA=IH :: DH : HE$; which, by projection, will become the above proportion.

149. *5th Cor. If there be any tangent and a double ordinate drawn from the point of contact, and also any line, parallel to the axis, and limited by the tangent and double ordinate; then shall the curve divide that line in the same ratio, as the line divides the double ordinate.* That is, $ig' : in :: cn : ne$. (fig. 27).

In the first place, let us remark that, in the projection, $co=on$, $in=ad$, $cm=me$; since the quantities marked with the same letters in fig. 26. are respectively equal.

Now, by similar triangles, we have da or $in : g'i :: ga : gi :: om : on-oc$; but, $oc = \frac{1}{2} cn$, $om = cm - co = \frac{1}{2} ce - \frac{1}{2} cn = \frac{1}{2} ne$; whence, by substitution, $in : g'i :: ne : cn$, as above.

150. *6th Cor.* If, from the point of contact C, two chords CE, CLG, be drawn and the two diameters EH and LK, then will GK be parallel to the tangent: for, CL and LG are proportional to CK and KE, and consequently to IL and LK, wherefore the triangles CIL, LKG, are similar, and GK is parallel to CH.

GENERAL THEOREM.

151. *The projection of a parabola is another parabola.* From this proposition we can deduce at once the following corollaries.

152. 1*st Cor. The squares of the ordinates applied to any diameter are as their abscisses.*

Fig. 29. For PM, $P'M'$, &c. are proportional to the ordinates of which they are the projections. The same thing being said of the abscisses, CP, CP', &c., we deduce from art. 130, that

$$CP : CP' :: PM^2 : P'M'^2.*$$

153. *2d Cor.* As AM, $A'M'$, &c. are equal to CP, CP', &c., and, because the preceding property is true, whatever may be the angle of the tangent and diameter CP, it follows that the lines AM, $A'M'$, &c. appended to the tangent, being moveable about their points of suspension, will always have their extremities in a parabola, whatever position the tangent may take round the point C.

154. *3d Cor. Two tangents to a parabola intersect in the diameter to which the chord, which joins their points of contact, is a double ordinate.*

For, the sub-tangent of each must be double their common abscisse.

155. *4th Cor. Any diameter bisects its double ordinates.*

Since, they may be considered, respectively, as the projections of the axis and its ordinates.

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* If a direct demonstration appeared necessary to prove, that a parabola is projected into another parabola, it might be inferred from this corollary, which then must be demonstrated *à priori*, by means of an operation nearly similar to that of Theorem V. and fig. 26.

GENERAL PROPOSITIONS ON CONIC SECTIONS AND TRANSVERSAL LINES.

THEOREM I.

156. *If, from a point taken in the axis of a conic section, there be a tangent and a secant drawn, and an ordinate let fall from the point of contact upon the axis, the two lines, that connect the foot of the ordinate to the points where the secant meets the curve, make equal angles with the ordinate, and conseqently also with the axis.*

Draw, to the base of the cone represented in figure 30, a tangent *pq* and a chord *mn*, both parallel to the diameter *di*; if, through these two lines and the vertex, a tangent and a cutting plane be drawn, they will intersect each other, at the vertex, in a horizontal line *VI*, *vi*, parallel to the lines *pq*, *mn*. The cutting plane, besides, will cut out of the cone the two elements *VM*, *vm*, and *VN*, *vn*.

157. Now, if the cone and the two planes be intersected by an oblique plane *IDd*, the two latter will be cut in two lines, which must meet at the point *d*, where their common line of intersection is cut by the oblique plane. One of these lines is tangent to the conic section, at the point *g* of the element (or ordinate) *vp*; the other is secant to the curve, and must evidently pass through the points *a* and *b* of the elements *vm*, *vn*, cut out of the cone.

158, Now, it is obvious, that in the circle the angles *bvg*, *avg*, are equal; consequently they are equal in the curve: and, because the lines of which *va* and *vb* are the projections, must, likewise be equally inclined to the ordinate projected vertically in *G*, it follows that the proposition is true for the conic section in its own plane.

159. This property will receive a direct demonstration in art. 176; but its generality may be made obvious now, by remarking that the demonstration would be exactly the same, if, instead of being a circle, the base of the cone was a conic section (fig 33) having its axis perpendicular to the horizontal trace of the cutting plane.

160. *Cor.* The triangles vrb, vsa, are similar, whence,
$rb : sa :: rv : sv :: bh : ha$, and, because $rb : sa :: bi : ai$, it will be $bh : ha :: bi : ai$.

By inclining the plane of the curve in any position whatever, its projection would be a conic section, its axis dci would become a diameter, and the perpendicular gg' an ordinate applied to it: and since the parts of the secant would preserve their ratio, the above proportion is the expression of the following general property.

161. *After having from any point drawn two tangents to a conic section, and a double ordinate through their points of contact, if any secant be drawn through the first point, the segments of the secant, between that point and the curve, will be proportional to the segments of the same contained between the curve and double ordinate.*

That is, as above, $AI : BI :: AH : BH$ (fig. 31).

162. 1*st. Cor.* By composition, we have:
$BI-AI : BH-AH :: AI : BI$, and, if C be the middle of the line BA, $BI-AI$ and $BH-AH$ will respectively be equal to $2\,CA$ and $2\,CH$; whence, $CA : CH :: AI : BI$.

163. 2*d Cor.* Again; $BI-AI : BH-AH :: BI+AI : BH+AH$, that is, $2\,CA : 2\,CH :: 2\,CI : 2\,CA$; whence, $CA^2 = CH \,.\, CI$: a very general property of conic sections, including the parabola and the circle, and of which arts. 64. and 104. are only particular cases.

164. 3*d Cor.* If we square the fundamental proportion, we get;

$$AI^2 : BI^2 :: AH^2 : BH^2; \text{ whence,}$$

$$BI^2-AI^2 : BH^2-AH^2 :: AH^2 : AI^2: \text{ but}$$

$$BI^2-AI^2=(BI-AI)\,(BI+AI)=2\,CA \,.\, 2\,CI, \text{ and}$$

$$BH^2-AH^2=(BH-AH)\,(BH+AH)=2\,CA \,.\, 2\,CH; \text{ whence,}$$

by substitution, and because $4\,CA$ is a common factor,

$$CI \cdot CH :: AH^2 : AI^2.$$

A general extension of arts. 66. and 109; which, as well as the preceding corollary, shows that many properties of the ellipse and hyperbola did not appear applicable to the parabola, because of their being demonstrated for particular cases.

THEOREM II.

165. *If, from any point, there be drawn two tangents, and two secants to a conic section, the two diagonals and also the two chords which join the points of in ersection of the secants, two and two, will intersect in the line which passes through the two points of contact of the tangents.*

Fig. 32. To the circular base *CBDA* of a cone, draw in any direction whatever the two parallel tangents *FA* and *FB*, and chords *GH*, *IK*, as well as the diagonals *HI*, *GK*, and the chords *HKN*, *GIN*; it is evident that these last diagonals and chords intersect in the diameter *AB*.

166. Conceive now, through the vertex of the cone, and each one of those lines, a plane. The planes passing through the tangents *FA*, *FB*, will be tangent, the others will be secant, planes. Now, intersect all the planes so drawn, as well as the cone, by an oblique plane; it will cut the cone in a curve, and each plane in a tangent, or secant, to it, according to the nature of the plane intersected.

167. But it is evident that, each one of the planes which pass through the vertex and the parallel lines *FB*, *IK*, *GH*, &c., must contain the straight line drawn through the vertex parallel to them: consequently, that line will be the common intersection of all such planes.

168. *The different intersections of these planes, by the oblique one, must therefore converge towards the point, where their common line is cut by the oblique plane,*

Consequently, *cbda* (fig. 34), being the curve cut out of the cone, the tangents *fb*, *fa*, and the secants *gh*, *ik*, which correspond to the parallel lines marked with the same letters in the circle (fig. 32), must pass through a common point *f*.

169. Again, each one of the points *g*, *i*, *k*, *h*, *m*, *n*, will be found where the line, which unites to the vertex of the cone its corresponding point in the circle, is intersected by the oblique plane. And the line *ab of the contacts* (that is to say, which joins the points of contact) will be the section made in the plane which passes through the vertex and the diameter *AB* (fig. 32).

Consequently, the chords hk, gi, produced, and also the diagonals gk, ih, intersect in the line ab of the contacts.*

THEOREM III.

170. *After having, from a fixed point, drawn to a conic section two tangents and any number of secants; if other tangents be drawn to the curve, at all the points where it is cut by the secants; these tangents will meet, two and two, in the right line which passes through the points of contact of the two first tangents.*

For, the tangents fa, fb, and secants fh, fd, fk, correspond to the parallel tangents and secants, marked with the same letters in the circle; and, in this latter, the two tangents IN', KN', intersect evidently in the diameter AB: wherefore, in the oblique section, (or perspective), the corresponding tangents in', kn', must meet in the line ab of the contacts; and, since the same thing might be said of any other two tangents gn'', hn'', the proposition appears manifest.

As to the tangents DE and CE, parallel to the diameter AB of the circle, they must, in the perspective, converge with it towards the same point (168): this point is e in fig. 34.

171. *Cor.* it is clear that, if, instead of a circle, we had assumed, for the base of the cone, any other conic section, and drawn, parallel to its axis, some tangents and secants, as is represented in fig. 33, the same things would have been remarked as in the circle. Fig. 33, (wherein the same letters designate the same points as in the circle) will render this perfectly obvious.

The circle is more convenient for such investigations; but then some attention is requisite, not to admit, as a general principle, a proposition which may be only a particular property of it.

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* When, as in the above example, cones and planes are drawn, from a fixed point, through the curves and right lines which compose a figure, the new figure, cut out of these cones and planes, is called the *perspective* of the former. Fig. 34 is the perspective of figs. 32 and 33.

The remark of art. 168. is the principle by which *vanishing points* are determined in PERSPECTIVE.

OF SOME GENERAL PROPERTIES OF CONIC SECTIONS, RESULTING FROM THE FOREGOING.

172. The two preceding theorems are readily applicable to the investigation of a great number of properties of conic sections: we will, exemplify it here upon some of the most useful.

It is first evident that Theorems II. and III. will be true, whatever may be the distance of the point f (fig. 34 and 35): if, then, this point be supposed at an infinite distance, the tangents fa, fb, will be parallel (fig. 35), the line ab will be a diameter, in which the chords cin, dkn, must intersect.

And, if the secant fcd pass through the centre o, then will $il=lk$; since $co=od$. That is; *Any diameter bisects its double ordinates, or parallels to its conjugate.*

173. Again; because the tangents at the points i and k must intersect in the line abn of the contacts, it follows that, the *two tangents at the extremities of a double ordinate intersect in its diameter.*

174. Fig. 36. After having drawn from a point f, the tangents fb, fa, and the diameter fcd, if a line gi be drawn parallel to the line ab of the contacts, it will be a double ordinate (172), and the secants fik, fgh, passing through its extremities, will meet the curve in two other points k and h, such that the line kh, which joins them, is also a double ordinate to the diameter cd: for, gi and kh produced must meet in the line abe of the contacts.

175. This understood; it is obvious that the diagonals gk, ih, must both pass through the foot p of the ordinate or line of the contacts; for, by art. 169, they cross each other in the line ab, and, because $il=gl$ and $kr=hr$, their meeting point must also be in the diameter fcd: whence the proportion,

$$gl \text{ or } il : kr :: lp : pr :: iv : vk: \quad \text{but} \quad il : kr :: fi : fk;$$

consequently, $fi : fk :: iv : vk$, which is the same proportion as in art. 161, whence the same corollaries might be obtained.

176. Again, if fcd were the axis of the curve, il, pb, and kr would be perpendicular to it; and then the angles ipl, kpr, would be equal. Therefore Theor. I. is a corollary of the preceding paragraph.

177. Farthermore, because the transversal lines id, kc intersect at the point m of the line ab (169), we have, by similar triangles;

$$il : ld :: mp : pd, \quad \text{and} \quad kr : rc :: mp : pc;$$

besides, $\quad il : cl :: np : pc, \quad$ and $\quad kr : rd :: np : pd,$ whence by multiplication; $\quad il^2 : ld \,.\, cl :: mp \,.\, np : pd \,.\, pc,$

and $\quad kr^2 : rc \,.\, rd :: mp \,.\, np : pc \,.\, pd,$

and by equality, $\quad il^2 : kr^2 :: ld \,.\, cl : rc \,.\, rd :$

that is, *the squares of the ordinates to any diameter are to each other as the rectangles of their abscisses.*

178 As this last demonstration does not suppose that the secant fcd is a diameter, it is evident, that the proposition would hold good for the parts of the ordinates il, kr, that would be intercepted by any secant whatever, drawn from the point f; and that, therefore, when so generalized, it also applies itself to the parabola.

This recapitulation of some of the principal properties of conic sections is a sufficient introduction to the general method of investigation, by which they might all be obtained: we will conclude it by the following remarkable proposition.

179. *If a right angle be supposed to move round a conic section, so that its sides may be constantly tangent to the curve, the vertex of the angle will describe the circumference of a circle concentric to the curve.*

Fig. 40. F and F' being the foci of the curve, and Ii, InI', $I'N'E$ three rectangular tangents to it, the perpendiculars NFN', Fn, $F'n'$, demitted upon them from the foci, will meet them in the points N, n, n', N', of the circle described on the transverse axis (47): consequently the arcs $n\,N$, $n'\,N'$, are equal; whence $I'n'=In$, and $I'n=In'$. Now, if we draw the tangent IT, we have $IT^2=In \,.\, In'=In \,.\, I'n=FN \,.\, FN'$: but, by a known property of the circle, $FN \,.\, FN'=FO \,.\, FO'$, a constant quantity. The tangent IT being then a constant quantity, the point I describes the circumference of a circle, the radius of which is CI.

The demonstration is the same for the hyperbola; and, in the parabola, the vertex of the tangential right angle describes the directrix (45, 140).

180. *Cor. The sum in the ellipse, and the difference in the hyperbola, of the squares of two conjugate diameters is a constant quantity.*

If the diameter iCb be drawn, it will bisect the double ordinate GG' (172); and the line BCI, drawn, through the centre C, pa-

rallel to GG', will be the conjugate of the first diameter iCb: but, from $AI : BI :: AH : BH$ (161), we have obtained $CA^2 = CH . CI$ (163); in like manner, we must have $Ca^2 = Ch . Ci$; and, because $G'h = Gh$, and EiI is a right angled triangle, $EC = CI = Ci$, whence, by similar triangles, hC or $HG = HI$, and consequently $hC + HC = CI$: with these substitutions, the sum of the above equations gives, for the ellipse, $CA^2 + Ca^2 = (hC + HC) CI = CI^2$, a constant quantity.

For the hyperbola, the sum, instead of the difference, of the equations must be taken.

181. *2d. Cor.* CTI being a right angled triangle, TI = *the semi-conjugate axis, which consequently is a mean proportional to FO and FO'* (60, 100).

THEOREM IV.

182. *If a trapezium be circumscribed about a conic section, its diagonals and the lines of the opposite contacts intersect at the same point; and the opposite sides of the inscribed trapezium, formed by the four points of contact, intersect in the diagonal contained between them.*

Fig. 39. That is, the diagonals ts and $n'n''$ intersect at the point o common to the lines ph and gl; and the chords gp, hl, produced meet in the diagonal $n''n'$ also produced; and, likewise, the chords pl, gh, meet at the point f of the diagonal ts.

This will appear evident, by considering fig. 39 as the perspective of fig. 33, which has the same points marked with the same letters (the circumscribed trapezium being the perspective of the square $SN'TN''$ (fig. 33.) tangent to the ellipse.)

For, all parallel lines will converge towards a common point; tangents will be the perspective of tangents; and the points of intersection of the different lines will correspond.*

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* In order to complete all the foregoing demonstrations, it may be necessary to prove that; *a conic section, and the secants which intersect it, can always be considered as the perspective of another conic section intersected by lines parallel to its axis*, which may be done, by showing, that it is always possible to find, for an inscribed trapezium, a rectangle, of which it may be the perspective: for it is evident that a rectangle inscribed in a conic section must have its sides parallel to the axis of the curve.

$DEFG$ (fig. 37.) being the given trapezium; on the line AB, drawn through

183. It may farther be added, that *the tangents to the points where one diagonal of the circumscribed trapezium cuts the curve, meet, in the other diagonal, at the same point where the two opposite sides of the inscribed trapezium intersect.* That is *ce*, *de*, meet at the point *e* of the diagonal *n″n′e*, as well as *gpe* and *hle*. The perspective of fig. 33, where these five lines are parallel, makes it evident.

THEOREM V.

184. *If, through any point taken within a conic section, any number of chords be drawn, and tangents be constructed at their extremities, the tangents for the same chord will meet in a point, and all such points will be in the same right line.*

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the points of intersection of its opposite sides produced, describe a semi-circle, which consider as standing in a vertical plane. Now, if a point *C* be assumed in that semi-circumference as the vertex of a pyramid, having the trapezium for its base, the lines *AC* and *BC* will each be the intersection, and also the trace, of two opposite faces of the pyramid; and any vertical plane *MN*, parallel to that of the semi-circle, will cut the faces *CBG*, *CBD*, in two lines *dq*, *gn*, parallel to the common trace *CB* of those faces: this same plane will, likewise, intersect the two other faces in two lines *mg*, *pf*, parallel to *AC*, and consequently perpendicular to the former lines. The rectangular figure *defg*, formed by those four lines, being the section made in the pyramid by the plane *MN*, will be the perspective of the trapezium.

As, in the last theorem, the perspective of a square has been used, it may not be amiss to determine, among the different positions of the vertex *C*, that *which makes the perspective of the trapezium a square*: in order to this, produce the diagonal *EG*, and, through the point *H* and the middle *O* of the lower semi-circle *AOB*, draw the line *OHC*, which determines the point *C*. For, the line *GH* being the vertical trace of the diagonal plane *CHG*, must be parallel to the perspective *gel* of the diagonal itself; and because the angles *ACO*, *OCB*, are equal, it follows that *mgl*=*lgn*, and, that *defg* is a square.

From this it is easy to conclude that, *a conic section can always be converted into a circle by perspective*: *DEFG* (fig. 38) being an inscribed trapezium, through the points *A* and *B*, where its opposite sides intersect, draw tangents to the curve; they will form a circumscribed trapezium, and the lines of their contacts must pass respectively through the points *A* and *B* (165) If now the circumscribed trapezium *defg* be made a square by perspective, *DEFG* will become a rectangle, and the curve itself, a circle: since it is impossible to inscribe a rectangle parallel to a circumscribed square in any other conic section than a circle.

If it be proposed to draw, through a given right line *fu* (fig. 39), two planes tangent to a sphere, assume in that line any number of points *f*, *v*, *e*, *u*, for the vertices of cones tangent to the sphere; their different circles of contact will all pass through the two points of contact of the plane and sphere (D. G. 192); and the line, which joins these points, will be the common intersection of all the planes of the circles of contact.

Now, a plane passed through the given line *fu* and the centre of the sphere will intersect this latter in a great circle, each of the cones in two tangents to that circle, and the bases of these cones in as many chords *ab*, *gl*, *cd*, *ph*, passing through the point *o*, where the common line of the circles of contact is intersected by the cutting plane.

The proposition is then manifest for the circle; and consequently for any conic section whatever. It does, with that of art. 170, constitute but one general property.

THEOREM VI.

185. *The three points of intersection of the opposite sides of any hexagon inscribed in a conic section, are situated in a right line.*

Fig. 41. *AEDBC* being the great circle of a sphere, and *AB*, *CD*, *EF*, the projections of any three other circles perpendicular to the plane of the figure, let it be required: *to find, on the surface of the sphere, a fourth circle tangent to the three given ones.*

It is evident that, if through the circles *AB* and *CD*, a cone be made to pass, (which is always possible*), and also a second cone

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* (A) This may be made evident as follows: let a trapezium *sn'tn''* (fig. 39) be circumscribed about a circle; draw its diagonals, *n'n''*, *st*; in the points *a* and *b* of the first diagonal, construct the tangents *fb*, *fa*, draw also the chords *lp*, *hg*, these four last lines will intersect in a common point *f* of the diagonal *st* produced (183, 186): in like manner, the chords *gp*, *hl*, will meet in the diagonal *n'n''*, produced; and, lastly, the transversal lines *ph*, *gl*, will intersect in the fixed point *o* common to the diagonals *n'n''* and *st* (182).

(B) This understood; consider the circle as the great section of a sphere; *gp* and *hl* as the projections of two small circles, through which it is proposed to draw a cone; and the points *f*, *t*, *s*, as the vertices of three cones tangent to the sphere; their bases will be the three circles, projected in *gp*, *ab*, *hl*, which

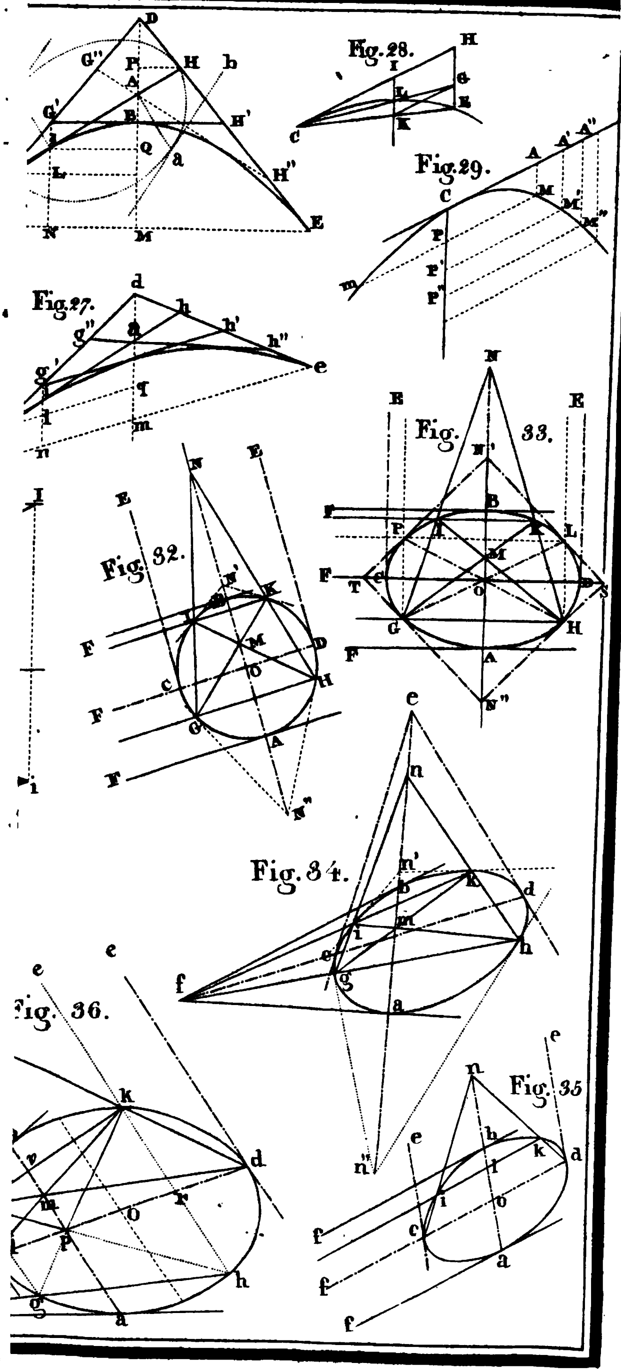
Fig. 27.
Fig. 28.
Fig. 29.
Fig. 32.
Fig. 33.
Fig. 34.
Fig. 35
Fig. 36.

through the circles AB and EF, a plane tangent to these two cones will cut out of the sphere a circle tangent to the three given ones. This same plane must also be tangent to the third cone, which would pass through the circles CD and EF, since this latter would lead to the same solution as the others. Consequently, the cutting plane will pass through the vertices G, H, I, of the three cones.

But there will be, below the diametral plane of the sphere, another circle tangent to the given ones, which must be obtained, as well as the first, by a plane tangent to the three cones. This second tangent plane must, likewise, contain the three vertices G, H, I, which are consequently in its line of intersection with the first.

The three vertices G, H, I, being then in a right line, if a section be made through this latter and the centre of the sphere, it will furnish in each cone two elements, and the parts of these elements, intercepted within the great circle cut out of the sphere, will form an inscribed hexagon $AEDBFC$. Whence the theorem is obvious for the circle, and, by perspective, for any conic section.

THEOREM VII.

186. *In any hexagon circumscribed about a conic section, the diagonals, drawn through its opposite angles, intersect in the same point.*

$ABCDEF$ (fig. 42), being the circumscribed hexagon, its six points of contact will be the summits of an inscribed one $abcdef$.

Now, the diagonal CF being drawn, the two tangents GT, GT',

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meeting at the common point e, the planes of the three bases will intersect in a common line projected in e.

(C) Now, if, in any direction whatever, a plane be made to pass through the line fts, it will cut out a section similar to that of the figure itself: that is to say, the sphere will be cut in a circle, the cones in tangents to that circle, and the planes of their bases in chords like gp, ab, hl, converging towards the point e, where the line common to those planes is intersected. The point f will then constantly be, as above, the converging point of all such chords as pl and gh, and the point o the invariable intersection of the transversal lines ph, gl (A). Consequently the points f and o are the vertices of two cones passing through the two circles projected in gp and hl.

in the points where it cuts the curve, must meet in the line *bcG* drawn through the points of contact (170) of the two adjacent sides *CbB*, *CcD*, of the circumscribed hexagon. These tangents must likewise meet in the line *fEG*, which joins the points of contact of the sides *Ff*, *Fe*.

The point *G*, therefore, is common to the two opposite sides of the inscribed hexagon and to the two tangents in the points *T*, *T'*, of the corresponding diagonal, in the circumscribed one.

In like manner, it might be proved that the tangents, corresponding to the diagonal *AD*, must meet at the point of intersection *H* of the opposite sides *dcH*, *faH*, of the inscribed hexagon.

And, the same thing being also true for the third diagonal *BE*, it follows: *that the three meeting points of the tangents, drawn to the two points of the curve situated in each diagonal, are the same as the converging points G, H, I, of the opposite sides of the inscribed hexagon.*

But (185) these points are in a straight line, and consequently (184) the chords, which join the points of contact of the tangents drawn from them, intersect in a common point. Whence, because those lines are also the diagonals of the circumscribed hexagon, the proposition is demonstrated.

187. *1st Cor.* If two adjacent sides *ED*, *DC*, approach, nearer and nearer, to being the same straight line; when they ultimately coincide, the vertex *D* of their angle will become their common point of contact (fig. 43). Consequently: *if, in any pentagon circumscribed about a conic section, and from two different angular points, two diagonals be drawn, and, if their point of intersection be joined to the fifth angle, the line of junction will pass through the point of contact of the opposite side.*

188. *2d Cor.* If two other sides are conceived to coincide, the pentagon will be reduced to a quadrilateral, and then, according to the sides which coincide, it will appear, either, as in art. 182; *that, the diagonals intersect in the line which joins two opposite points of contact* (fig. 44).

189. Or, that: *if a diagonal be drawn through two angles, the two lines, drawn from the other angles to the opposite points of contact, intersect in that diagonal* (fig. 45).

190. *3d Cor.* Lastly, if the polygon is, in the same manner, re-

duced to a triangle, it will be seen; that, *the three lines which unite its vertices to its points of contact intersect in the same point.*

These corollaries may be proved *a priori* by perspective. The foregoing are the most useful of the numerous propositions relating to transversal lines. They furnish for conic sections a great many very simple and convenient constructions.

PROBLEMS.

CONSTRUCTIONS OF CONIC SECTIONS BY A CONTINUOUS MOTION.*

PROBLEM I. *Having given the axes of an ellipse or hyperbola, to construct the curve.*

ELLIPSE. From the extremity b (fig. 47) of the conjugate, with a radius equal to the semi-transverse, describe an arc, which will cut the transverse axis in the foci F and f (31), in which fix, by two pins, the two ends of a thread equal in length to the transverse Ee. Then carry a pencil round by the thread, keeping it always stretched; this pencil will trace out the curve. For, in any position, $Fg+gf=Ee$ (30).

Otherwise. On any point D, with a radius equal to the difference DB of the axes, describe an arc; then, from the point B where it cuts the conjugate axis, and through the point D, draw the line Ba, which make equal to the semi-transverse: then will a be a point of the curve. For, through it, draw the ordinate FaA to intersect the circumference described on the axis; then, by similar triangles, the semi-conjugate $aD : AC :: Fa : FA$ (57).

In practice, an instrument called a *trammel* is used to describe the curve by a continuous motion: in the rod aB, the point B is a fixed pin, D is a moveable one, and a is a moveable nut made to hold a pencil: then, the distances aD and aB being made respec-

* Five conditions are necessary to determine a conic section, such as five points, four points and a tangent, &c. the centre answers for two conditions.

tively equal to the semi-conjugate and semi-transverse, while the points B and D move in the groves Ee and Bb, the extremity a describes the curve.

Sometimes, and particularly for small drawings, three points on the edge of a small strip of paper are used instead of the trammel.

HYPERBOLA. From the two axes we know how to determine the foci (100). This done, fix the extremity of a rule AB (fig. 48) at the focus B; and then fasten, to the second focus C and to the other extremity A of the rule, a string ADC equal to the difference between the rule and transverse axis. Now, if the point A be moved round the centre B, the angle D of the string ADC, by keeping it always tight and close to the rule AB, will describe one branch of the curve. For, the difference of BD and DC will be equal to the constant difference of BDA and ADC (30).

PROBLEM II. *Having given the focus and directrix of a parabola, to construct it.*

If a thread, equal in length to BC (fig. 49), be fixed at the extremity C of a square ABC, and if the side AB of the square be moved along the right line AD, the point E being kept close to the side of the square, and the string tight, this point will describe the curve: since $EF=BE$ in every position (131).

The foregoing constructions, though generally known, are very little used because of their obvious want of accuracy. It is probably owing to the difficulty of tracing out conic sections, that they are not oftener employed. On that account, we would specially recommend the following method as being very ready in practice, and particularly on a large scale.

PROBLEM III. *General construction by means of a moving triangle.*

Let a right angle FGH (figs. 50, 51) be moved round the focus F, so that, one of its legs FG passing constantly through this focus, its vertex G may move in the circumference described on the transverse axis. Its second leg GH will, in every position of the right angle, be tangent to the conic section (47), which will be rapidly and accurately delineated, by drawing a sufficient number of the tangents determined by the successive positions of the second leg GH of the right angle.

In the parabola (fig. 49), the vertex G of the angle describes the right line tangent at the vertex of the curve (48).

In the ellipse, the operation requires only the arc of the circle

intercepted by the tangents parallel to the transverse axis; and, in the hyperbola, the arc included between the asymptotes. This will appear evident in performing the operation.

CONSTRUCTIONS OF CONIC SECTIONS BY POINTS.

PROBLEM IV. *Having given the chord of an arc of a conic section, and the diameter to that chord, to construct the arc of the curve.*

Through the extremity of the chord, draw two parallels AC, BD (fig. 52, 53, 54) to the diameter, and limited by the tangent at the extremity of the same. Then divide, both these parallels and each half chord AG, GB, into an equal number of equal parts (into four for ex.) This done, join the points of division of the parallels AC, BD, to the extremity F of the diameter, by as many lines; and draw some other lines through the other extremity E of the same diameter and the points of division of the chord: these last lines will intersect the corresponding lines of the first system in as many points of the curve.

Pins may then be fixed at each point, and a slip being bent round them, the curve may be traced by it.

This problem is the same as: *having given a diameter and its double ordinate to construct the curve.*

In the *parabola*, (fig. 54), the lines 11, 22, &c. are parallel, since all its diameters are infinite.

Demonstration. We have by similar triangles,

$$mo : Fm :: n3 : Fn = \tfrac{3}{4}DB = \tfrac{3}{4}FG,$$

$$\text{and} \quad mo : mE :: G3 : GE;$$

whence, by multiplication, and because $n3 = GB$ and $G3 = \tfrac{3}{4}GB$,

$$mo^2 : Fm \, . \, mE :: GB^2 : FG \, . \, GE \quad (12).$$

The demonstration is nearly the same for the parabola.

Other construction for the parabola.

Make, on the given diameter, DF equal to FG, and draw the lines AD and BD; they must be tangent to the curve (54). Having obtained these two lines, divide them each into a same number of equal parts, and join the opposite points of division: the lines so drawn will all be tangent to the same parabola; (147) and will, when taken in sufficient number, show accurately the shape of the curve.

Cor. The above construction for the ellipse will answer when two conjugate diameters are given.

PROBLEM V. *In general, when two conjugate diameters of a conic section are given,* the curve may be constructed by making them perpendicular to each other; constructing a curve of which these perpendicular lines would be the axes; and then inclining all the ordinates under the original angle of the diameters (82, 124).

PROBLEM VI. *Having given a point and the asymptotes of a hyperbola, to construct it.* See Art. 113, 118.

PROBLEM VII. *Having given two conjugate diameters of a hyperbola, to describe it.*

Fig. 48. at the extremity *D* of one of the diameters, and bisected by it, draw a line $E'De'$ equal and parallel to its conjugate; then the lines OE', Oe', will be the asymptotes (114) and *D* a point of the curve; which reduces the problem to the preceding.

PROBLEM VIII. *To construct a conic section similar to another.*

Let us suppose that the ratio of their linear dimensions is that of 1 to 2: from any point *F* draw any number of lines *FB*, *FL*, *FH*, &c. bisect them at *b*, *l*, *h*, &c.: or else, after having obtained the first point *b*, draw the chords *bl*, *lh*, &c. successively parallel to *BL*, *LH*, &c. and all the points so obtained will be in the new curve (119).

If there be no reason to take the point *F* in a particular position, it will be preferable to take it in the centre of the curve.

The second curve might also be obtained by bisecting the ordinates and abscisses of the given one.

DRAWING OF TANGENTS TO CONIC SECTIONS.

Arts. 35 and 40 show the construction of tangents by means of their directrices.

Art. 41, 42, 43, 44, 49, when the foci are known.

Art. 63, 64, give some constructions for the ellipse.

Art. 108 relates to the hyperbola.

And Art. 134, 135, 136, 138, 139, 141 contains constructions of tangents to the parabola.

PROBLEM IX. *To draw to a conic section a tangent parallel to a given line.*

Let PR (fig. 49, 50, 51) be the given line, draw to it, and through the focus f, a perpendicular fP; it intersects the circle described on the tranverse axis in a point g, through which drawing a parallel to the given line PR, it must be the tangent required.

If the centre and focus were not known, draw two chords parallel to the given line and bisect them by a line which must pass through the point of contact.

CONSTRUCTIONS BY MEANS OF TRANSVERSAL LINES.

PROBLEM X. *To draw a tangent to a conic section from a point given without the curve.*

From the given point F (fig. 56), draw any two secants FL, FH; join the points where they intersect the curve by the two chords hPE, HLE, and also by the two diagonals hL, PH; connect the point of intersection E of the chords to that O of the diagonals by the line EOB' which will determine the two points of contact B and B'.

PROBLEM XI. *To draw a tangent to a conic section, when the point of contact is given.*

Assume any five points in the curve and join them by five lines, 1, 2, 3, 4, 5, (fig. 59) and let 6 denote the point of contact given. Find the intersection E of 2 and 5 produced and that G of 1 and 4; draw through them the line FEG, and join the point F, where it intersects the chord 3 produced, to the point 6; and $F6$ will be the tangent sought.

For, the point of contact 6 may be considered as an evanescent chord; and, as such, it may be taken for the sixth side of a hexagon, in which, therefore, it must intersect its opposite side in the right line FEG (185).

These two constructions require but the use of a ruler. For the last one it is not necessary that the curve should be described, it will serve *to find at once five tangents to five given points of an undescribed conic section.*

PROBLEM XII. *Having given five points of a conic section, to construct the curve.*

Fig. 41. *C*, *A*, *E*, *D*, *B*, being the five given points; produce *AC* and *DB* to their point of intersection *G*; produce also *AE*; and, from the point *B*, draw any line *BFH*. Join the point *H*, where it cuts *AE*, with the point *G*. Then, through the point *I* where the last side *DE* of the inscribed pentagon intersects the line *GH*; draw the line *ICF*; and the point *F* will be in the curve (185). any number of points might be found in the same way.

PROBLEM XIII. *Having given five tangents to a conic section, to construct the curve.*

The five tangents form a circumscribed pentagon, in which draw the diagonals *EB*, *FC* (fig. 43), and then the line *AOD* which will determine the point of contact *D* of the tangent *EC*. Find, in the same manner, the other points of contact; and proceed as in Problem XII.

2d. *Construction. By means of the circumscribed pentagon, any number of tangents may be obtained*, which will, if in sufficient number, give a very accurate description of the curve.

In the diagonal *EB* (fig. 57), take any point *O*, and through it draw the lines *DO*, *CO*; they will cut the opposite tangents *EF*, *AB*, in two points *F* and *A*; and the line *FA* is tangent to the unknown curve.

For, since the diagonals *AD*, *FC*, *BE*, pass through the same point *O*, the line *FA* is the sixth side of a hexagon circumscribed about a conic section (186). In like manner, any number of tangents might be constructed.

PROBLEM XIV. *Having given four tangents and a point of contact, to construct the curve.*

B (fig. 45) being the given point of contact and *AC*, *CE*, *EF*, *FA*, the four tangents, draw the diagonal *EA* and the line *FB*; then, through the point *O* where they intersect, pass the line *COD*; and the point *D* will be the point of contact of the line *FA* (189), in the same way the other points of contact might be found. The rest of the construction as in the followin . problem.

PROBLEM XV. *Having given a circumscribed triangle and two points of contact, to construct the curve.*

D and *G* (fig. 58) being the points of contact, draw *DB* and *EG*, and, through their intersection *O*, pass the line *HOC* which determines the third contact *C* of the triangle (190).

Now, in the line DG, assume any point M, and through it draw the lines EMA, BMF; the line FA will be a fourth tangent to the curve, the point of contact f of which may be found by drawing the line CMf. For, the line FA must be the fourth side of a circumscribed trapezium, since the diagonals and lines of the opposite contacts intersect in the same point M (182).

Problem xvi. *Having given two tangents, their two points of contact and a third point of the curve, to construct it.*

Fig. 39. gv, and vl being the given tangents, g, l, their points of contact, and h the third given point, draw through v any line fvu; produce hl to e and hg to f; then draw fpl and gpe and their point of intersection p belongs to the curve. For, the points of intersection of the opposite sides of an inscribed trapezium are situated in a right line, which contains the points of intersection of the opposite sides of the circumscribed trapezium (182).

Any number of other points may be found by changing the direction of the line fvu: and, if the tangents at the points p and h were required, they may be determined, by drawing, through the point e and the point o (where the diagonals intersect) the line $en'on''$, which cuts vl and vg in the points n', n'', through which and the point u the tangent must pass (183).

Problem xvii. *To describe a conic section to touch a right line in a given point, and to pass through three other given points.*

Consider, as in Problem XI. the point of contact f (fig. 59) as an evanescent chord: A, D, C, being the given points, produce DC to its intersection F with the given tangent; through f draw any line BfG; find its intersection G with the line DA produced; join FG and then produce Af to E: now the line EC, which unites the points E and C, will cut the assumed line BG in a point B of the curve. For, it will complete an inscribed hexagon of which the tangent is the sixth side (185 and Problem XI).

All the preceding solutions have the advantage to require but a ruler for their construction. The doctrine of transversal lines is not, however, confined to that kind of solutions; it may also be usefully employed to determine the centre and other parts of conic sections. Of this we will present only one example here.

Problem xviii. *Having given five points, to find the centre &c. of the curve.*

E, F, A, B, C, (fig. 60.) being the given points, let it be required

to find the diameter to the double ordinate AB. Produce the lines FE, BC, to their intersection H, through which draw GH parallel to AB. If GH be considered as the line which contains the points of intersection of the opposite sides of an inscribed hexagon, it is evident that the side ED, opposite to AB, must be parallel to it; since they must intersect in their common parallel GH.

Draw then EDI parallel to AB and GH, produce AF to G; and join CG, which intersects EI in the point D of the curve. Now, bisect ED and AB; and om will be the required diameter: or else, what is easier, find the point of intersection of the chords AE, BD, as also that of the diagonals DA, BE; and join them by a line which must be the diameter (172, 175).

A similar construction in another direction would give another diameter, and therefore the centre, &c. &c.

Problem xix. *To determine the centre of a conic section, from some given tangents.*

Fig. 43. Bisect the line ef, which joins the points of contact of the tangents Fe, Ff; bisect also the chord fa, and draw the lines Fm, Am, which intersect at the centre (172).

PROBLEMS RELATIVE TO OBLIQUE ORDINATES AND DIAMETERS.

In almost every case, those problems may be solved by making the oblique ordinates perpendicular to their axis, and finding the solution as if intended for the altered curve: and then restoring the result to its real position, by inclining the ordinates under their original angle.

The truth of this method will be understood, by considering the given curve as an oblique projection of the altered one, and both consequently as belonging to the same cylinder. Problem V. affords an example of it, which is, in fact, about the only useful problem of that kind.

The only important question which remains to be solved is: *to determine the axes of a conic section.*

When the curve is drawn, describe on its centre a circle, which will intersect the curve in four points, which join by two chords; and the line which bisects these chords will be one of the axes, since it divides equally two parallel chords of the circle, to which it is consequently perpendicular.

PROBLEM XX. *Having given two conjugate diameters, to find the axes, without drawing any part of the curve.*

ELLIPSE. Through the extremity D (fig. 61) of the diameter DC, draw a parallel EF to the other diameter AB; erect upon it the perpendicular Do equal to the semi-diameter AO; and, upon o with the radius oD, describe a circle $agDb$. It is evident, that this circle, being raised round the line EF, may be considered as a circular section of a cylinder having the ellipse for its base. For, if the horizontal diameter ab be drawn, the lines aA, oO, bB, will be parallel, and may be considered as the oblique lines which project the circle $agDb$. This understood, since conjugate diameters of the ellipse are the projections of rectangular diameters of the circle; it remains to find in the circle the rectangular diameters which project themselves at right angles.

But the vertex of their angle is at o, in the circle, and is projected at O, in the ellipse: whence it follows, that a circle, having its centre in EF, and passing through those two points, will determine the corresponding diameters both in the circle and in the ellipse.

Draw then to the middle of oO the perpendicular Nl; and on N describe the circle $EoFO$; and EO, FO will be the directions of the two axes. Their lengths will be obtained by remarking, that they are the projections of the rectangular radii og, ol; and that, consequently, the parallel elements gG, lL, of the imaginary cylinder must determine their extremities G and L.

The curve can now be constructed by any of the known methods, or, by drawing any radius oin, and finding its oblique projection OIn.

In like manner, *any pair of conjugate diameters* OI, OK, *might be deduced from the corresponding diameters* oi, ok, *of the circle.*

HYPERBOLA. Find, as in Problem VII, the two asymptotes CB, CF (fig. 62); and bisect their angle to obtain the direction CO of the transverse. Then, through the extremity A of the given diameter, draw, to the transverse, the perpendicular EAF; and describe on it the semi-circumference EaF, the ordinate of which Aa is equal to the semi-conjugate.

For, CO being considered as the axis, C the vertex, and EaF the base, of a cone. If A be a point of a hyperbola cut out by a plane parallel to the diametral one, it is well known that the dis-

tance Aa of that point from the vertical plane is the semi-conjugate (93).

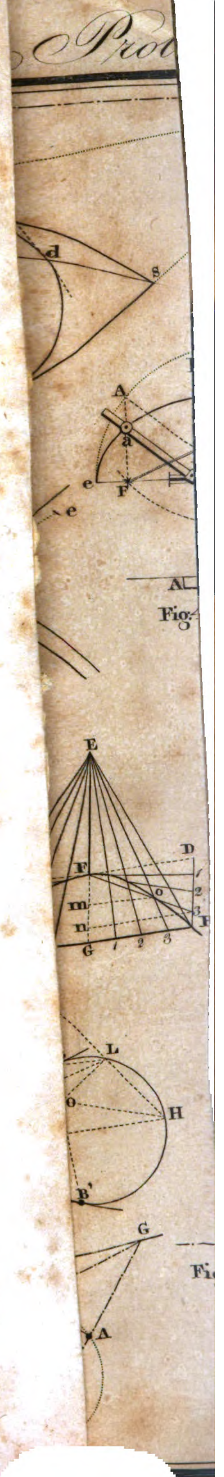
The length of the transverse will be found by drawing to it a perpendicular Gh, such that it may be equal to Aa or OH.

PARABOLA. *Having given a diameter and a double ordinate, to find the axis, &c. of the curve.*

Let AM be the diameter, and DC the double ordinate; make $AB=BM$, and draw the lines AD, DC; they must be tangent to the curve (154): now, draw DE and CE, so that the angles ADE, ACE, may be respectively equal to DAM, and CAM; their point of intersection E will be the focus (134), and, consequently, VE the axis of the parabola.

Otherwise, draw the normals DI, CH, and the perpendicular ordinates DG, CF; and construct the line VFI parallel to the given diameter, so that FG may be equal to HI. VFI will be the axis: for, FH will evidently be equal to GI (141).

THE END OF CONIC SECTIONS.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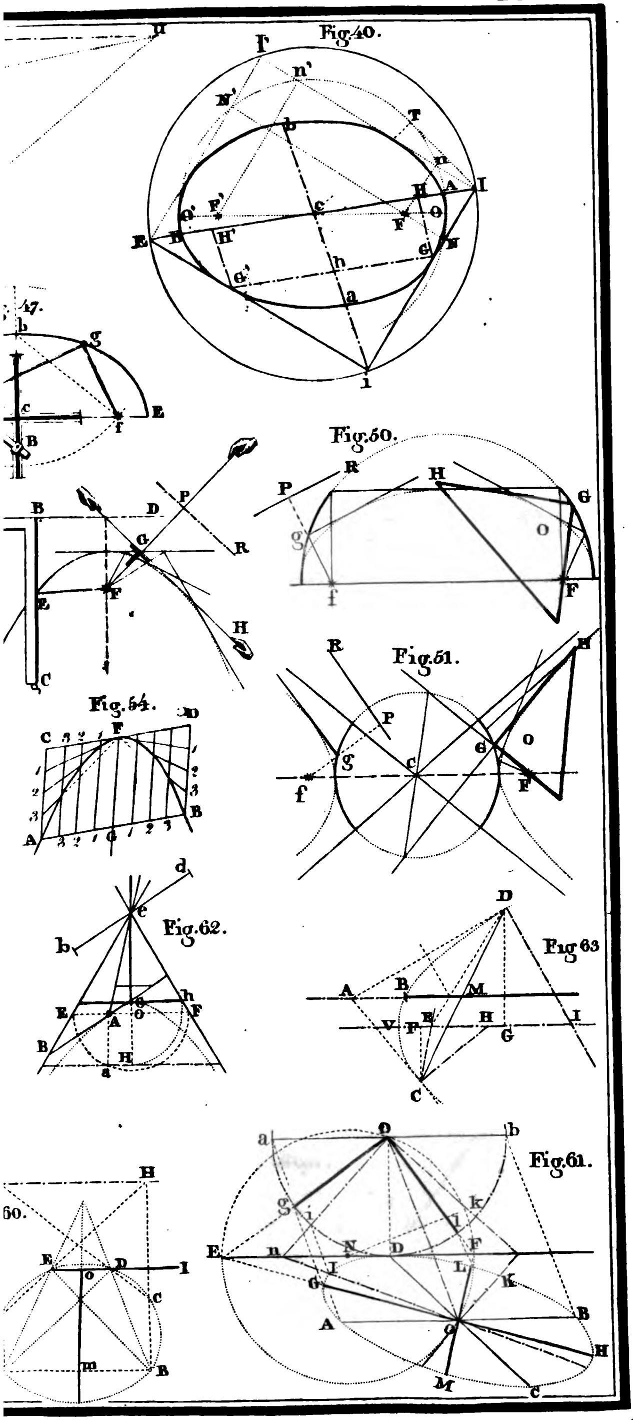
Fig.40.
47.
Fig.50.
Fig.51.
Fig.54.
Fig.62.
Fig 63
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Fig.61.